

Trace Construction of a Basis for the Solution Space of sl_N qKZ Equation

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Abstract

The trace of intertwining operators over the level one irreducible highest weight modules of the quantum affine algebra of type A_{N-1} is studied. It is proved that the trace function gives a basis of the solution space of the qKZ equation at a generic level. The highest-highest matrix element of the composition of intertwining operators is explicitly calculated. The integral formula for the trace is presented.

1 Introduction

In this paper we study solutions of the quantized Knizhnik-Zamolodchikov (qKZ) equation associated with the quantum group $U_q(sl_N)$. The idea in this paper stems from the study of solvable lattice models.

The qKZ equation was introduced in [6] as the equation satisfied by the highest-highest matrix element of the intertwining operators of quantum affine algebra. For generic values of parameters the set of matrix elements give a basis of the solution space over the field of appropriate periodic functions. The connection matrix of two solutions with different asymptotics have been calculated from the commutation relation of intertwining operators.

The solutions of the qKZ equation associated with $U_q(sl_2)$ is systematically studied by Tarasov and Varchenko [15] (see also references in it). In [15] the solutions are described as the multidimensional q-hypergeometric integrals. It is proved that, for generic values of parameters, the q-hypergeometric solutions give a basis of the solution space over the field of

appropriate periodic functions. The connection matrix is determined as the representation of Felder's elliptic quantum group.

In this paper we propose another description of the basis of the solution space of the qKZ equation as the traces of intertwining operators of quantum affine algebra.

Let us consider the $U_q(\mathfrak{sl}_N)$ modules V_1, \dots, V_n and the trigonometric R matrix $R_{ij}(z_i/z_j)$ acting on the tensor product $V_i \otimes V_j$. The qKZ equation is the q difference equation for the $V_1 \otimes \dots \otimes V_n$ valued function $f(z_1, \dots, z_n)$ of the form

$$\begin{aligned} f(\dots, pz_j, \dots) &= R_{jj-1}(pz_j/z_{j-1}) \cdots R_{j1}(pz_j/z_1)(\kappa^{-H})_j \\ &\quad \times R_{jn}(z_j/z_n) \cdots R_{jj+1}(z_j/z_{j+1})f(z_1, \dots, z_n), \end{aligned} \quad (1)$$

where $\kappa^{-H} = \prod_{i=1}^{N-1} \kappa_i^{-h_i}$, h_1, \dots, h_{N-1} is a basis of the Cartan subalgebra of \mathfrak{sl}_N and $(\kappa^{-H})_j$ means that κ^{-H} acts on V_j . The complex numbers p and κ_i 's are the parameters of the equation. If we write $p = q^{2(k+N)}$ the number k is called level.

Let Λ_i ($0 \leq i \leq N-1$) be the fundamental weights of $\widehat{\mathfrak{sl}_N}$. We identify Λ_i ($1 \leq i \leq N-1$) with the fundamental weights of \mathfrak{sl}_N . In this paper we consider the case where all V_i are isomorphic to the N dimensional irreducible module V with the highest weight Λ_1 or Λ_{N-1} .

Let $V(\Lambda_i)$ be the irreducible highest weight $U_q(\widehat{\mathfrak{sl}_N})$ module with the highest weight Λ_i and V_ζ the evaluation module of V . Then there exist, up to normalization, unique intertwining operators $\Phi(\zeta)$ and $\Psi^*(\xi)$:

$$\Phi(\zeta) : V(\Lambda_{i+1}) \longrightarrow V(\Lambda_i) \otimes V_\zeta, \quad \Psi^*(\xi) : V_\xi \otimes V(\Lambda_i) \longrightarrow V(\Lambda_{i+1}).$$

We extend the index i of Λ_i to the set of integers and read it by modulo N . The operators $\Phi(\zeta)$ and $\Psi^*(\xi)$ are sometimes called of type I and type II respectively [8]. The difference between type I and type II is in the place where the evaluation module is. For type I it is on the right of the highest weight module while for type II it is on the left.

Denote by D the grading operator of the principal gradation of $V(\Lambda_i)$ and consider the following trace:

$$\begin{aligned} G(\zeta_1, \dots, \zeta_m | \xi_1, \dots, \xi_n | x, \kappa) &= \\ F(\zeta | \xi | x)^{-1} \sum_{i=0}^{N-1} \text{tr}_{V(\Lambda_i)} \left(x^D \kappa^H \Phi(\zeta_1) \cdots \Phi(\zeta_m) \Psi^*(\xi_n) \cdots \Psi^*(\xi_1) \right) \end{aligned} \quad (2)$$

which is a function taking the value in $\text{Hom}_{\mathbf{C}}(V^{\otimes n}, V^{\otimes m})$. Here $F(\zeta | \xi | x)$ is some scalar function (cf. (15)). By the commutation relation of the intertwining operators, the cyclic property of the trace and the functional equation of $F(\zeta | \xi | x)$, this function satisfies the equations:

$$\begin{aligned} G(\zeta | \dots x \xi_i \cdots | x, y) &= G(\zeta | \xi | x, \kappa) \bar{R}_{ii+1}(\xi_i/\xi_{i+1}) \cdots \bar{R}_{in}(\xi_i/\xi_n)(\kappa^{-H})_{\xi_i} \\ &\quad \times \bar{R}_{i1}(x \xi_i/\xi_1) \cdots \bar{R}_{ii-1}(x \xi_i/\xi_{i-1}), \end{aligned} \quad (3)$$

$$\begin{aligned} G(\dots x^{-1} \zeta_i \cdots | \xi | x, \kappa) &= \bar{R}_{jj-1}(x^{-1} \zeta_j/\zeta_{j-1}) \cdots \bar{R}_{j1}(x^{-1} \zeta_j/\zeta_1)(\kappa^{-H})_{\zeta_i} \\ &\quad \times \bar{R}_{jm}(\zeta_j/\zeta_m) \cdots \bar{R}_{jj+1}(\zeta_j/\zeta_{j+1}) G(\zeta | \xi | x, \kappa), \end{aligned} \quad (4)$$

where $\bar{R}(\zeta)$ is the trigonometric R matrix (c.f. (7)), $\bar{R}_{ii+1}(\xi_i/\xi_{i+1})$ acts non-trivially on $V_{\xi_i} \otimes V_{\xi_{i+1}}$ in $V^{\otimes n}$ etc. The equation (4) has precisely the same form as the qKZ equation (1). Let ${}^t G$

be the transpose of G , that is, ${}^tG \in \text{Hom}_{\mathbf{C}}(V^{*\otimes m}, V^{*\otimes n})$, V^* being the dual vector space of V . Then, as the equation for tG , (3) is of the same form as (1). Since we use the principal gradation in this paper, to make a precise correspondence between the parameter x and the parameter p in (1) we need to consider G as a function of $z_j = \zeta_j^N$ and $u_j = \xi_j^N$. Then if $x^N = p = q^{2(k+N)}$, tG and G satisfy the qKZ equation of level k and level $-k - 2N$ in the variables u and z respectively.

In this paper, if $x^{-N} = q^{2(k+N)}$, we say (4) the qKZ equation of level k with the value in $V^{\otimes m}$.

Let \mathcal{S}_k^n and \mathcal{S}_k^{*n} be the space of meromorphic solutions of the qKZ equation of level k with the value in $V^{\otimes n}$ and $V^{*\otimes n}$ respectively and \mathcal{F} the field of x periodic meromorphic functions in n variables. Then the function G defines two maps simultaneously:

$${}^tG(\zeta | \cdot | x, \kappa) : V^{*\otimes m} \otimes \mathcal{F} \longrightarrow \mathcal{S}_k^{*n}, \quad (5)$$

$$G(\cdot | \xi | x, \kappa) : V^{\otimes n} \otimes \mathcal{F} \longrightarrow \mathcal{S}_{-k-2N}^m. \quad (6)$$

In (5), ζ_1, \dots, ζ_m are parameters of the map and in (6), ξ_1, \dots, ξ_n are parameters of the map. We consider the case $n = m$. We assume $|x| < 1$. We shall prove that, if x and κ are generic, (5) is an isomorphism for the generic values of ζ_1, \dots, ζ_m and (6) is an isomorphism for the generic values of ξ_1, \dots, ξ_n . It is proved by showing that the determinant of G does not vanish identically. We calculate the determinant at $x = 0$ where G reduces to the highest-highest matrix element. For the level one irreducible module $V(\Lambda_i)$ the matrix elements can be calculated explicitly without integral. This is expected because at $q = 1$ we have such formula calculated by using the Frenkel-Kac bosonization of $V(\Lambda_i)$ [5]. In the case of $U_q(\widehat{sl}_2)$ the formula is given in [8]. For $U_q(\widehat{sl}_N)$ we carry out the integral of the integral formula given by the Frenkel-Jing bosonization in [10] in a similar manner to $N = 2$ case.

The case $x = q^2$ is relevant to the physical quantities in solvable lattice models. In fact at this value of x if we further specialize the variables ζ_i and ξ_j appropriately, the trace functions give correlation functions and form factors of the solvable lattice model constructed from the R-matrix $\bar{R}(\zeta)$. We have calculated the determinant of G for $N = n = 2$ and $x = q^2$ explicitly. By the q series expansion we checked that $\det G$ does not vanish identically for $n = 3$. We conjecture that the determinant does not vanish identically at $x = q^2$. This suggests that the trace description can be effective for the completeness problem of the space of local fields [13][1].

The bosonization of intertwining operators makes it possible not only to derive the integral formula for the matrix elements but also to derive the integral formula for the trace. We have given the integral formula. Therefore the integral formula for the basis of the solution space of (3) and (4) is given.

The plan of this paper is as follows.

In section 2 we give necessary notations of quantum affine algebra of type $A_{N-1}^{(1)}$. We introduce the intertwining operators for the level one integrable modules in the principal picture in section 3. In section 4 we give the relation between principal picture and homogeneous picture. It serves for translating the results in the references into principal picture and vice versa. In section 5 we introduce the trace of intertwining operators and derive the equations satisfied by them. The main results and their proof is given in section 6. In section 7 we give an example of the concrete expression of the determinant of the trace of intertwining operators

in the case of $U_q(\widehat{sl_2})$. In section 8 we give the integral formula for the matrix element of the intertwining operators. The integrated formula for the matrix element is given in section 9. In section 10 the integral formula of the trace of intertwining operators is presented. In appendix A we refer the integral formula for the trace of intertwining operators of $U_q(\widehat{sl_2})$ in [8], since in this case it is possible to simplify the formula a bit. This simplification is used in the calculation in the example of section 7. The bosonic expression of the intertwining operators are reviewed in appendix B. The list of the expression of the operators in terms of their normal ordered operators is given in appendix C. In appendix D a derivation of the integral formula for the trace of intertwining operators is explained.

2 Preliminary

Let $A = (a_{ij})$ be the generalized Cartan matrix of type $A_{N-1}^{(1)}$, $\{\alpha_i\}_{i=0}^{N-1}$ and $\{h_i\}_{i=0}^{N-1}$ the set of simple roots and simple coroots respectively so that $\langle \alpha_i, h_j \rangle = a_{ij}$.

The quantum affine algebra $U'_q(\widehat{sl_N})$ is the Hopf algebra generated by $e_i, f_i, t_i^{\pm 1}$ ($0 \leq i \leq N-1$) with the following defining relations:

$$t_i t_j = t_j t_i, \quad t_i^{\pm 1} t_i^{\mp 1} = 1, \quad t_i e_j t_i^{-1} = q^{\langle h_i, \alpha_j \rangle} e_j, \quad t_i f_j t_i^{-1} = q^{-\langle h_i, \alpha_j \rangle} f_j,$$

$$[e_i, f_j] = \delta_{ij} \frac{t_i - t_i^{-1}}{q - q^{-1}}, \quad \sum_{k=0}^{N-1} (-1)^k e_i^{(k)} e_j^{(1-a_{ij}-k)} = \sum_{k=0}^{1-a_{ij}} (-1)^k f_i^{(k)} f_j^{(1-a_{ij}-k)} = 0 \quad i \neq j,$$

where $e^{(k)} = e^k/[k]!$ and similarly for $f^{(k)}$, $[k]! = [k] \cdots [2][1]$, $[k] = (q^k - q^{-k})/(q - q^{-1})$.

The coproduct Δ and the antipode S are given by $\Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i$, $\Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i$, $\Delta(t_i) = t_i \otimes t_i$ and $S(e_i) = -t_i^{-1} e_i$, $S(f_i) = -f_i t_i$, $S(t_i) = t_i^{-1}$.

We extend the algebra $U'_q(\widehat{sl_N})$ by adding the element D such that

$$[D, e_i] = e_i, \quad [D, f_i] = -f_i, \quad [D, t_i^{\pm 1}] = 0, \quad \Delta(D) = D \otimes 1 + 1 \otimes D.$$

The resulting algebra is denoted by $U_q(\widehat{sl_N})$. We say that an element $X \in U_q(\widehat{sl_N})$ is of degree n if $[D, X] = nX$. We denote by Λ_i ($0 \leq i \leq N-1$) the fundamental weights of $\widehat{sl_N}$. We identify Λ_i ($1 \leq i \leq N-1$) with the fundamental weights of sl_N . We extend the index i of Λ_i to any integer and read it by modulo N .

For a highest weight $U_q(\widehat{sl_N})$ module M with the highest weight vector v , D defines a grading on M by $D(Xv) = nXv$ for a degree n element X in $U_q(\widehat{sl_N})$. The evaluation representation $V_\zeta = \bigoplus_{j=0}^{N-1} \mathbb{C} v_j$ of $U'_q(\widehat{sl_N})$ associated with the irreducible $U_q(sl_N)$ module with the highest weight Λ_1 is given by

$$f_i v_j = \zeta^{-1} \delta_{ij+1} v_{j+1}, \quad e_i v_j = \zeta \delta_{ij} v_{j-1}, \quad t_i v_j = q^{-\delta_{ij} + \delta_{ij+1}} v_j,$$

where the index of v_j should be read modulo N . In terms of $\{\Lambda_j\}$ the weight of v_j , which we denote $\text{wt} v_j$, is given by $\text{wt} v_j = \Lambda_{j+1} - \Lambda_j$.

We denote the binomial coefficient by ${}_n C_r$, that is, $(1+x)^n = \sum_{r=0}^n {}_n C_r x^r$.

In this paper two kinds of variables appear, one is u and z , the other is ξ and ζ . They are always related by the relation $u = \xi^N$ and $z = \zeta^N$ except in Appendix A where $u = -\xi^2$ and $z = \zeta^2$.

3 Intertwining operators

In [2][10] the evaluation representation, R matrices and intertwining operators are described in the homogeneous grading. We shall rewrite them to the principal picture so that the description is consistent with the sl_2 case in [8] and that the equations for the trace of intertwining operators are free from cumbersome factors.

Let $P\bar{R}(\zeta_1/\zeta_2)$ be the intertwiner from $V_{\zeta_1} \otimes V_{\zeta_2}$ to $V_{\zeta_2} \otimes V_{\zeta_1}$ normalized as $P\bar{R}(\zeta_1/\zeta_2)(v_0 \otimes v_0) = v_0 \otimes v_0$, where P is the permutation operator, $P(v \otimes w) = w \otimes v$. We define the components of $\bar{R}(\zeta)$ by $\bar{R}(\zeta)(v_i \otimes v_j) = \sum_{i',j'} \bar{R}(\zeta)_{i'j'}^{ij} v_{i'} \otimes v_{j'}$.

They are given by (cf. [2])

$$\begin{aligned} \bar{R}(\zeta)_{jj}^{jj} &= 1, \quad \bar{R}(\zeta)_{jk}^{jk} = b(\zeta) = \frac{q(1 - \zeta^N)}{1 - q^2 \zeta^N} \quad (j \neq k), \\ \bar{R}(\zeta)_{kj}^{jk} &= c_{jk}(\zeta) = \frac{1 - q^2}{1 - q^2 \zeta^N} \zeta^{N\theta(k-j)+j-k} \quad (j \neq k), \end{aligned} \quad (7)$$

where $\theta(k) = 1$ ($k \geq 0$), $= 0$ (otherwise) and $0 \leq j, k \leq N - 1$.

Let $V(\Lambda_i)$ be the irreducible highest weight $U_q(\widehat{\mathfrak{sl}_N})$ module with the highest weight Λ_i and the highest weight vector $|\Lambda_i\rangle$, $V(\Lambda_i)^*$ the restricted dual right highest weight module of $V(\Lambda_i)$ with the highest weight vector $\langle \Lambda_i|$ such that $\langle \Lambda_i|, |\Lambda_i\rangle = 1$, where \langle, \rangle is the dual pairing. We denote $\langle \Lambda_j|, X|\Lambda_i\rangle = \langle \Lambda_j|X, |\Lambda_i\rangle = \langle \Lambda_j|X|\Lambda_i\rangle$ for any $X \in \text{Hom}_{\mathbf{C}}(V(\Lambda_i), V(\Lambda_j))$, where X acts on $V(\Lambda_j)^*$ from the right.

The type I and type II intertwining operators $\Phi^{(i)}(\zeta)$ and $\Psi^{*(i)}(\xi)$ are the $U'_q(\widehat{\mathfrak{sl}_N})$ linear operators of the form

$$\begin{aligned} \Phi^{(i)}(\zeta) : V(\Lambda_{i+1}) &\longrightarrow V(\Lambda_i) \otimes V_\zeta, \quad \Phi^{(i)}(\zeta) = \sum_{\epsilon=0}^{N-1} \Phi_\epsilon^{(i)}(\zeta) \otimes v_\epsilon, \\ \Psi^{*(i)}(\xi) : V_\xi \otimes V(\Lambda_i) &\longrightarrow V(\Lambda_{i+1}), \quad \Psi^{*(i)}(\xi)(v_\mu \otimes u) = \Psi_\mu^{*(i)}(\xi)u. \end{aligned}$$

In the second equation $u \in V(\Lambda_i)$. We normalize them by the condition that

$$\langle \Lambda_i| \Phi_i^{(i)}(\zeta) |\Lambda_{i+1}\rangle = 1, \quad \langle \Lambda_{i+1}| \Psi_i^{*(i)}(\xi) |\Lambda_i\rangle = 1.$$

Under these normalization the operators $\Phi^{(i)}(\zeta)$ and $\Psi^{*(i)}(\xi)$ are unique.

We sometimes omit the upper index (i) of $\Phi^{(i)}(\zeta)$ and $\Psi^{*(i)}(\xi)$ for the sake of simplicity. They satisfy the following commutation relations ([2]):

$$R(\zeta_1/\zeta_2)\Phi(\zeta_1)\Phi(\zeta_2) = \Phi(\zeta_2)\Phi(\zeta_1), \quad (8)$$

$$\Psi^*(\xi_2)\Psi^*(\xi_1)R^*(\xi_1/\xi_2) = \Psi^*(\xi_1)\Psi^*(\xi_2), \quad (9)$$

$$\Phi(\zeta)\Psi^*(\xi) = \tau(\zeta/\xi)\Psi^*(\xi)\Phi(\zeta), \quad (10)$$

where

$$\tau(\zeta) = \zeta^{1-N} \frac{\theta_{q^{2N}}(q\zeta^N)}{\theta_{q^{2N}}(q\zeta^{-N})}$$

and for any complex number p such that $|p| < 1$ we set $\theta_p(z) = (z; p)_\infty (pz^{-1}; p)_\infty (p; p)_\infty$, $(z; p)_\infty = \prod_{k=0}^{\infty} (1 - p^k z)$.

The matrices $R(\zeta)$ and $R^*(\zeta)$ is given by $R(\zeta) = r(\zeta)\bar{R}(\zeta)$ and $R^*(\zeta) = r^*(\zeta)\bar{R}(\zeta)$ with

$$r(\zeta) = \zeta^{-1} \frac{(q^{2N} z^{-1}; q^{2N})_\infty (q^2 z; q^{2N})_\infty}{(q^{2N} z; q^{2N})_\infty (q^2 z^{-1}; q^{2N})_\infty}, \quad r^*(\zeta) = -\zeta^{-1} \frac{(q^{2N} z^{-1}; q^{2N})_\infty (q^{2N-2} z; q^{2N})_\infty}{(q^{2N} z; q^{2N})_\infty (q^{2N-2} z^{-1}; q^{2N})_\infty}.$$

In these commutation relations we use the following notation: for $v_i \otimes v_j \in V_{\zeta_1} \otimes V_{\zeta_2}$ and $v_{j'} \otimes v_{i'} \in V_{\zeta_2} \otimes V_{\zeta_1}$, the equation $v_i \otimes v_j = v_{j'} \otimes v_{i'}$ means $v_i = v_{i'}$ and $v_j = v_{j'}$. This is for the sake of simplifying the description of the equation. Thus in terms of components (8), (9) and (10) are written as

$$R(\zeta_1/\zeta_2)_{\epsilon'_1 \epsilon'_2}^{\epsilon'_1 \epsilon'_2} \Phi_{\epsilon'_1}(\zeta_1) \Phi_{\epsilon'_2}(\zeta_2) = \Phi_{\epsilon_2}(\zeta_2) \Phi_{\epsilon_1}(\zeta_1), \quad (11)$$

$$R^*(\xi_1/\xi_2)_{\epsilon'_1 \epsilon'_2}^{\epsilon_1 \epsilon_2} \Psi_{\epsilon'_2}^*(\xi_2) \Psi_{\epsilon'_1}^*(\xi_1) = \Psi_{\epsilon_1}^*(\xi_1) \Psi_{\epsilon_2}^*(\xi_2), \quad (12)$$

$$\Phi_\epsilon(\zeta) \Psi_\mu^*(\xi) = \tau(\zeta/\xi) \Psi_\mu^*(\xi) \Phi_\epsilon(\zeta). \quad (13)$$

Let σ be the automorphism of $U_q(\widehat{\mathfrak{sl}}_N)$ induced by the Dynkin diagram automorphism, $\sigma(e_i) = e_{i+1}$, $\sigma(f_i) = f_{i+1}$, $\sigma(h_i) = h_{i+1}$. The indices are understood by modulo N . The Dynkin automorphism σ induces the linear automorphism of V_ζ , the linear isomorphism between the left highest weight modules $V(\Lambda_i)$ and $V(\Lambda_{i+1})$, the linear isomorphism between the right highest weight modules $V(\Lambda_i)^*$ and $V(\Lambda_{i+1})^*$ by $\sigma(v_j) = v_{j+1}$, $\sigma(|\Lambda_i \rangle) = |\Lambda_{i+1} \rangle$, $\sigma(\langle \Lambda_i |) = \langle \Lambda_{i-1} |$ with the properties $\sigma(Xv) = \sigma(X)\sigma(v)$ and $\sigma(v^*X) = \sigma(v^*)\sigma^{-1}(X)$ for $X \in U_q(\widehat{\mathfrak{sl}}_N)$, $v \in V(\Lambda_i)$, $v^* \in V(\Lambda_i)^*$.

Then the intertwining operators satisfy the following relations

$$\Phi^{(i)}(\zeta) = (\sigma \otimes \sigma) \Phi^{(i-1)}(\zeta) \sigma^{-1}, \quad \Psi^{*(i)}(\xi) = \sigma \Psi^{*(i-1)}(\xi) (\sigma^{-1} \otimes \sigma^{-1}).$$

In terms of components these are

$$\Phi_\epsilon^{(i)}(\zeta) = \sigma \Phi_{\epsilon-1}^{(i-1)}(\zeta) \sigma^{-1}, \quad \Psi_\mu^{*(i)}(\xi) = \sigma \Psi_{\mu-1}^{*(i-1)}(\xi) \sigma^{-1}.$$

These relations are proved by checking the intertwining properties and the normalization conditions of the right hand side of the equations using the relation

$$(\sigma \otimes \sigma) \Delta = \Delta \sigma.$$

The R matrix $\bar{R}(\zeta)$ is also invariant with respect to σ :

$$\bar{R}(\zeta)_{\sigma(i')\sigma(j')}^{\sigma(i)\sigma(j)} = \bar{R}(\zeta)_{i'j'}^{ij},$$

where $\sigma(i) = i+1$ ($0 \leq i \leq N-2$), $\sigma(N-1) = 0$.

4 Principal-homogeneous correspondence

We shall give relations between the intertwining operators in this paper and those in [10][2].

Let $V_z^{(h)} = \bigoplus_{j=0}^{N-1} \mathbf{C} v_j$ be the homogeneous evaluation representation given by

$$f_i v_j = z^{-\delta_{i0}} \delta_{ij+1} v_{j+1}, \quad e_i v_j = z^{\delta_{i0}} \delta_{ij} v_{j-1}, \quad t_i v_j = q^{-\delta_{ij} + \delta_{ij+1}} v_j,$$

The map $V_\zeta \longrightarrow V_z^{(h)}$ given by $v_i \mapsto v_i \zeta^i$ commutes with the action of $U'_q(\widehat{\mathfrak{sl}_N})$, where $z = \zeta^N$.

Let $\tilde{\Phi}_{\Lambda_{i+1}}^{\Lambda_i V}(z)$ and $\tilde{\Phi}_{\Lambda_i}^{V^* \Lambda_{i+1}}(z)$ be the intertwining operators in [10]:

$$\tilde{\Phi}_{\Lambda_{i+1}}^{\Lambda_i V}(z) : V(\Lambda_{i+1}) \longrightarrow V(\Lambda_i) \otimes V_z^{(h)}, \quad \tilde{\Phi}_{\Lambda_i}^{V^* \Lambda_{i+1}}(z) : V(\Lambda_i) \longrightarrow V_z^{(h)*} \otimes V(\Lambda_{i+1}).$$

We set

$$\tilde{\Phi}^{h(i)}(z) = \tilde{\Phi}_{\Lambda_{i+1}}^{\Lambda_i V}(z), \quad \tilde{\Phi}_{\Lambda_i}^{V^* \Lambda_{i+1}}(z) = \sum_{j=0}^{N-1} v_j^* \otimes \tilde{\Psi}_j^{*h(i)}(z),$$

where $\{v_j^*\}$ is the dual basis to $\{v_j\}$, $\langle v_i, v_j^* \rangle = \delta_{ij}$. Then

$$\Phi_j^{(i)}(\zeta) = \zeta^{i-j} \tilde{\Phi}_j^{h(i)}(\zeta^N), \quad \Psi_j^{*(i)}(\zeta) = \zeta^{j-i} \tilde{\Psi}_j^{*h(i)}(\zeta^N),$$

where $0 \leq i, j \leq N-1$. We remark that the dual representation V^* in $\tilde{\Phi}_{\Lambda_i}^{V^* \Lambda_{i+1}}(z)$ in [10] is with respect to the antipode inverse.

Let $\bar{R}^{(h)}(z) = \bar{R}_{V(1)V(1)}(z)$ be the R matrix in [2]. Then

$$\bar{R}(\zeta)_{i'j'}^{ij} = \bar{R}^{(h)}(\zeta^N)_{i'j'}^{ij} \zeta^{i-i'}.$$

5 Trace of intertwining operators

In order to appropriately normalize the trace of intertwining operators we first introduce scalar functions which satisfies some functional equations. For complex numbers p_1, \dots, p_k such that $|p_i| < 1$ for any i , we define

$$(z; p_1, \dots, p_k)_\infty = \prod_{r_1, \dots, r_k=0}^{\infty} (1 - p^{r_1} \dots p^{r_k} z).$$

We set $\{z\} = (z; q^{2N}, x^N)_\infty$ and

$$h^{(\sigma)}(z|x) = \frac{\{q^{1+\sigma} x^N z^{-1}\} \{q^{1+\sigma} z\}}{\{q^{2N-1+\sigma} x^N z^{-1}\} \{q^{2N-1+\sigma} z\}}, \quad (14)$$

where $\sigma = 0, \pm 1$. Let us define

$$\bar{F}(\mathbf{z}|\mathbf{u}|x) = \prod_{a < b} h^{(+)}\left(\frac{z_b}{z_a} | x\right) \left(\prod_{a,b} h^{(0)}\left(\frac{u_a}{z_b} | x\right) \right)^{-1} \prod_{a < b} h^{(-)}\left(\frac{u_a}{u_b} | x\right), \quad (15)$$

and

$$F(\zeta|\xi|x) = \bar{F}(\mathbf{z}|\mathbf{u}|x) \left(\frac{\prod_{a,b} \theta_x(-\xi_b/\zeta_a)}{\prod_{a < b} \theta_x(-\zeta_b/\zeta_a) \prod_{a < b} \theta_x(-\xi_a/\xi_b)} \right)^{N-1}.$$

The function F satisfies the following equations:

$$\begin{aligned} F(\cdots, \zeta_{j+1}, \zeta_j, \cdots | \xi | x) &= r(\zeta_j/\zeta_{j+1}) F(\zeta | \xi | x), \\ F(\zeta | \cdots, \xi_{j+1}, \xi_j, \cdots | x) &= r^*(\xi_j/\xi_{j+1}) F(\zeta | \xi | x), \\ F(x^{-1}\zeta_1, \cdots, \zeta_m | \xi | x) &= \prod_{j=1}^n \tau(\xi_j/\zeta_1) F(\zeta_2, \cdots, \zeta_m, \zeta_1 | \xi | x), \\ F(\zeta | x\xi_1, \cdots, \xi_n | x) &= \prod_{j=1}^m \tau(\xi_1/\zeta_j) F(\zeta | \xi_2, \cdots, \xi_n, \xi_1 | x). \end{aligned}$$

For complex numbers y_1, \dots, y_{N-1} let us set $y^{\pm H} = \prod_{j=1}^{N-1} y_j^{\pm h_j}$. Let x be a complex number satisfying $|x| < 1$. We define the normalized trace function as

$$G^{(i)}(\zeta | \xi | x, y) = \frac{\text{tr}_{V(\Lambda_i)}(x^D y^H \Phi(\zeta_1) \cdots \Phi(\zeta_m) \Psi^*(\xi_n) \cdots \Psi^*(\xi_1))}{F(\zeta | \xi | x)}. \quad (16)$$

and set

$$G(\zeta | \xi | x, \mathbf{y}) = \sum_{i=0}^{N-1} G^{(i)}(\zeta | \xi | x, \mathbf{y}). \quad (17)$$

These functions take the value in $\text{Hom}_{\mathbf{C}}(V^{\otimes n}, V^{\otimes m})$. We define the components of G by

$$G(\zeta | \xi | x, y)(v_{\mu_1} \otimes \cdots \otimes v_{\mu_n}) = \sum_{\epsilon_1, \dots, \epsilon_m} G(\zeta | \xi | x, y)_{\mu_1, \dots, \mu_n}^{\epsilon_1, \dots, \epsilon_m} v_{\epsilon_1} \otimes \cdots \otimes v_{\epsilon_m}.$$

By the functional equation of F and the commutation relations of intertwining operators the function G satisfies the following system of equations:

$$\begin{aligned} \bar{R}_{ii+1}(\zeta_i/\zeta_{i+1}) G(\cdots \zeta_i \zeta_{i+1} \cdots | \xi | x, y) &= G(\cdots \zeta_{i+1} \zeta_i \cdots | \xi | x, y), \\ G(\zeta | \cdots \xi_i \xi_{i+1} \cdots | x, y) \bar{R}_{ii+1}(\xi_i/\xi_{i+1}) &= G(\zeta | \cdots \xi_{i+1} \xi_i \cdots | x, y), \\ G(x^{-1}\zeta_1, \zeta_2, \cdots, \zeta_m | \xi | x, y) &= (y^{-H})_{\zeta_1} G(\zeta_2, \cdots, \zeta_m, \zeta_1 | \xi | x, y), \\ G(\zeta | x\xi_1, \xi_2, \cdots, \xi_n | x, y) &= G(\zeta | \xi_2, \cdots, \xi_n, \xi_1 | x, y) (y^{-H})_{\xi_1}, \end{aligned}$$

where $\bar{R}_{ij}(\zeta_i/\zeta_j)$ acts nontrivially on the component $V_{\zeta_i} \otimes V_{\zeta_j}$ in $V_{\zeta_1} \otimes \cdots \otimes V_{\zeta_m}$, $\bar{R}_{ij}(\xi_i/\xi_j)$ acts nontrivially on the component $V_{\xi_i} \otimes V_{\xi_j}$ in $V_{\xi_1} \otimes \cdots \otimes V_{\xi_n}$ and $(y^{-H})_{\zeta_1}$ means that y^{-H} acts on the component V_{ζ_1} etc. In thses equations we use the same notation as in the equation (8), (9) and (10) to avoid the use of permutation operators in the equations (see the comment before (11), (12), (13)).

As a consequence of these equations G satisfies

$$G(\cdots x^{-1}\zeta_i \cdots |\xi|x, y) = K_i^{(1)}(\zeta_1, \cdots, \zeta_m|x, y)G(\zeta|\xi|x, y), \quad (18)$$

$$G(\zeta|\cdots x\xi_i \cdots |x, y) = G(\zeta|\xi|x, y)K_i^{(2)}(\xi_1, \cdots, \xi_n|x, y) \quad (19)$$

$$\begin{aligned} K_i^{(1)}(\zeta_1, \cdots, \zeta_m|x, y) &= \\ \bar{R}_{ii-1}(x^{-1}\zeta_i/\zeta_{i-1}) \cdots \bar{R}_{i1}(x^{-1}\zeta_i/\zeta_1)(y^{-H})_{\zeta_i} \bar{R}_{im}(\zeta_i/\zeta_m) \cdots \bar{R}_{ii+1}(\zeta_i/\zeta_{i+1}), \\ K_i^{(2)}(\xi_1, \cdots, \xi_n|x, y) &= \\ \bar{R}_{ii+1}(\xi_i/\xi_{i+1}) \cdots \bar{R}_{in}(\xi_i/\xi_n)(y^{-H})_{\xi_i} \bar{R}_{i1}(x\xi_i/\xi_1) \cdots \bar{R}_{ii-1}(x\xi_i/\xi_{i-1}). \end{aligned}$$

Note that

$${}^t K_i^{(2)}(\zeta_1, \cdots, \zeta_m|x, y) = K_i^{(1)}(\zeta_1, \cdots, \zeta_m|x^{-1}, y).$$

If we denote $x^N = q^{2(k+N)}$ then the corresponding equations (18) and the transpose of (19) are the qKZ equations of level $-k - 2N$ and level k respectively.

From the Dynkin symmetry of the intertwining operators, $G^{(i)}$ and G satisfy the following equations:

$$\begin{aligned} G^{(i+1)}(\zeta|\xi|x, y_1, \cdots, y_{N-1})_{\sigma(\mu_1), \dots, \sigma(\mu_n)}^{\sigma(\epsilon_1), \dots, \sigma(\epsilon_m)} &= y_1 G^{(i)}(\zeta|\xi|x, y_1^{-1}y_2, \cdots, y_1^{-1}y_{N-1}, y_1^{-1})_{\mu_1, \dots, \mu_n}^{\epsilon_1, \dots, \epsilon_m}, \\ G(\zeta|\xi|x, y_1, \cdots, y_{N-1}) &= y_1 G(\zeta|\xi|x, y_1^{-1}y_2, \cdots, y_1^{-1}y_{N-1}, y_1^{-1}). \end{aligned}$$

We define $\bar{G}^{(i)}(\zeta|\xi|x, y)$ by the similar formula to (16) where $F(\zeta|\xi|x)$ is replaced by $\bar{F}(\mathbf{z}|\mathbf{u}|x)$. Then define the function $\bar{G}(\zeta|\xi|x, \mathbf{y})$ similarly to (17).

The function \bar{G} satisfies the following system of equations:

$$\left(\frac{\zeta_{i+1}}{\zeta_i}\right)^{N-1} \bar{R}_{ii+1}(\zeta_i/\zeta_{i+1}) \bar{G}(\cdots \zeta_i \zeta_{i+1} \cdots |\xi|x, y) = \bar{G}(\cdots \zeta_{i+1} \zeta_i \cdots |\xi|x, y), \quad (20)$$

$$\bar{G}(\zeta|\cdots \xi_i \xi_{i+1} \cdots |x, y) \left(\frac{\xi_i}{\xi_{i+1}}\right)^{N-1} \bar{R}_{ii+1}(\xi_i/\xi_{i+1}) = \bar{G}(\zeta|\cdots \xi_{i+1} \xi_i \cdots |x, y), \quad (21)$$

$$\bar{G}(x^{-1}\zeta_1, \zeta_2, \cdots, \zeta_m|\xi|x, y) = \bar{G}(\zeta_2, \cdots, \zeta_m, \zeta_1|\xi|x, y) \prod_{j=1}^n \left(\frac{\zeta_1}{\xi_j}\right)^{N-1},$$

$$\bar{G}(\zeta|x\xi_1, \xi_2, \cdots, \xi_n|x, y) = \bar{G}(\zeta|\xi_2, \cdots, \xi_n, \xi_1|x, y) \prod_{j=1}^n \left(\frac{\zeta_j}{\xi_1}\right)^{N-1}.$$

For the Dynkin diagram symmetries exactly the same equation as $G^{(i)}$ and G holds for $\bar{G}^{(i)}$ and \bar{G} .

Up to now we do not mention to in which sense the trace (16) exists and to the validity of the application of the commutation relations (8), (9) and (10) inside the trace. By definition the trace (16) exists as a formal power series in x whose coefficient is a finite sum of matrix elements of the intertwining operator $\Phi(\zeta_1) \cdots \Psi^*(\xi_1)$. It is known that the latter matrix element, which originally defined as a series in ζ and ξ , are analytically continued to give a meromorphic function on $(\mathbf{C}^*)^{n+m}$, where $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$ is the algebraic torus. In fact, as we show in section 10, the series in x can be summed up explicitly to express the trace (16) as a meromorphic function on $(\mathbf{C}^*)^{n+m}$. Since the commutation relations (8), (9) and (10) hold in the sense of analytically continued matrix element, they are applicable to the series expression of the trace in x and hence to the meromorphic expression of the trace.

6 Completeness of trace function

In this section we assume $n = m$ and study the determinant of \bar{G} . Let $(V^{\otimes n})_{k_0, \dots, k_{N-1}} = \{v \in V^{\otimes n} | t_i v = q^{k_{i-1} - k_i} v\}$ be the weight subspace of $V^{\otimes n}$ with respect to $U_q(\mathfrak{sl}_N)$. From the definition of the trace and the intertwining operators, \bar{G} commutes with the action of t_i ($1 \leq i \leq N-1$). Therefore the determinant of \bar{G} is the product of the determinants taken at each weight subspace. In the following we fix a set of numbers k_0, \dots, k_{N-1} . The determinant always means the determinant taken at the weight subspace $(V^{\otimes n})_{k_0, \dots, k_{N-1}}$.

Theorem 1 *We assume $|x| < 1$. Then $\det \bar{G}(\zeta | \xi | x, y)$ does not vanish identically as a function of ζ_i 's, ξ_j 's, x, q and y_k 's. Moreover $\det \bar{G}(\zeta | \xi | x, 1)$ does not vanish identically. Here $y = 1$ means that all $y_i = 1$.*

We say that $(x, q, y) = (x, q, y_1, \dots, y_{N-1})$ is generic if $\det \bar{G}(\zeta | \xi | x, y)$ does not vanish identically as a function of ζ 's and ξ 's. For fixed (x, q, y) we say that the set of complex numbers ζ_1, \dots, ζ_n is generic if $\det \bar{G}(\zeta | \xi | x, y)$ does not vanish identically as a function of ξ 's. Similarly for fixed (x, q, y) we say that the set of complex numbers ξ_1, \dots, ξ_n is generic if $\det \bar{G}(\zeta | \xi | x, y)$ does not vanish identically as a function of ζ 's.

Corollary 1 *Suppose that (x, q, y) is generic. The map (5) is an isomorphism for generic values of ζ_1, \dots, ζ_n and the map (6) is an isomorphism for generic values of ξ_1, \dots, ξ_n .*

To prove Theorem 1 we first express the determinant of \bar{G} using some single function. Let us set

$$f(\zeta | \xi | x, \mathbf{y}) = \bar{G}(\zeta | \xi | x, \mathbf{y})_{0^{k_0} \dots (N-1)^{k_{N-1}}}^{0^{k_0} \dots (N-1)^{k_{N-1}}},$$

where $(0^{k_0} \dots (N-1)^{k_{N-1}})$ means the $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ such that $\epsilon_1 \leq \dots \leq \epsilon_n$ and the number of i in ϵ is k_i . We call f the extremal component of \bar{G} .

From (20) and (21) we have, for $l > k$,

$$\begin{aligned} \bar{G}(\dots \zeta_i \zeta_{i+1} \dots | \xi | x, y)_{\mu}^{\dots lk \dots} &= a_{kl}^{(3)}(\zeta_i / \zeta_{i+1}) \bar{G}(\dots \zeta_i \zeta_{i+1} \dots | \xi | x, y)_{\mu}^{\dots kl \dots} \\ &\quad + a_1(\zeta_i / \zeta_{i+1}) \bar{G}(\dots \zeta_{i+1} \zeta_i \dots | \xi | x, y)_{\mu}^{\dots kl \dots}, \end{aligned} \quad (22)$$

$$\begin{aligned} \bar{G}(\zeta | \dots \xi_i \xi_{i+1} \dots | x, y)_{\dots lk \dots}^{\epsilon} &= a_{kl}^{(4)}(\xi_i / \xi_{i+1}) \bar{G}(\zeta | \dots \xi_i \xi_{i+1} \dots | x, y)_{\dots kl \dots}^{\epsilon} \\ &\quad + a_2(\xi_i / \xi_{i+1}) \bar{G}(\zeta | \dots \xi_{i+1} \xi_i \dots | x, y)_{\dots kl \dots}^{\epsilon}, \end{aligned} \quad (23)$$

where

$$\begin{aligned} a_1(\zeta) &= \frac{q^{-1} \zeta^{N-1} (1 - q^2 \zeta^N)}{1 - \zeta^N}, \quad a_2(\zeta) = \frac{q^{-1} \zeta^{-N+1} (1 - q^2 \zeta^N)}{1 - \zeta^N}, \\ a_{kl}^{(3)}(\zeta) &= -\frac{q^{-1} \zeta^{N+k-l} (1 - q^2)}{1 - \zeta^N}, \quad a_{kl}^{(4)}(\zeta) = -\frac{q^{-1} \zeta^{l-k} (1 - q^2)}{1 - \zeta^N}. \end{aligned}$$

Using these equations it is possible to express any component of \bar{G} in terms of f . In order to describe this expression precisely let us introduce the lexicographical order, on the set of $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ such that $\#\{j | \epsilon_j = i\} = k_i$, comparing from left to right. The minimal element is $(0^{k_0} \dots (N-1)^{k_{N-1}})$. It is convenient to associate $M = (M_0, \dots, M_{N-1})$ with $\epsilon = (\epsilon_1, \dots, \epsilon_n)$

such that $M_i = \{m_1^{(i)} < \dots < m_{k_i}^{(i)}\} = \{j | \epsilon_j = i\}$. We use both notations to specify components of \bar{G} . We denote the minimal element by $M^0 = (M_0^0, \dots, M_{N-1}^0)$. For $M = (M_0, \dots, M_{N-1})$ we set $(\zeta_{M_0}, \dots, \zeta_{M_{N-1}}) = (\zeta_{m_1^{(0)}}, \dots, \zeta_{m_{k_{N-1}}^{(N-1)}})$. Then

$$\bar{G}(\zeta|\xi|x, y)_L^M = \sum_{M' \leq M} a^{MM'} \mathbf{G}(\zeta_{M'_0}, \dots, \zeta_{M'_{N-1}} | \xi|x, y)_L^{M^0},$$

with

$$a^{MM} = \prod_{r>l} \prod_{a \in M_r, b \in M_l, a < b} a_1(\zeta_a / \zeta_b).$$

Similarly

$$\bar{G}(\zeta|\xi|x, y)_L^M = \sum_{L' \leq L} b^{LL'} \mathbf{G}(\zeta|\xi_{L'_0}, \dots, \xi_{L'_{N-1}} | x, y)_{M^0}^M,$$

with

$$b^{LL} = \prod_{r>l} \prod_{a \in L_r, b \in L_l, a < b} a_2(\xi_a / \xi_b).$$

These equations mean that the matrix \bar{G} and the matrix whose components consist of f with permuted variables are connected by the product of two triangular matrices with diagonal components $(a^{MM})_M$ and $(b^{MM})_M$ respectively. Thus we have

Proposition 1

$$\begin{aligned} \det(\bar{G}(\zeta|\xi|x, y)_\mu^\epsilon) &= \left(\prod_{i < j} a_1(\zeta_i / \zeta_j) a_2(\xi_i / \xi_j) \right)^{n(k_0, \dots, k_{N-1})} \\ &\times \det(f(\zeta_{M_0}, \dots, \zeta_{M_{N-1}} | \xi_{L_0}, \dots, \xi_{L_{N-1}} | x, y))_{M, L}, \end{aligned}$$

where

$$n(k_0, \dots, k_{N-1}) = \sum_{0 \leq l < r \leq N-1} n_{l,r}(k_0, \dots, k_{N-1}),$$

$$n_{l,r}(k_0, \dots, k_{N-1}) = \prod_{j=0}^{N-2} n_{-2-\sum_{i=0}^{j-1} k'_i} C_{k'_j},$$

$$(k'_0, \dots, k'_{N-1}) = (k_0, \dots, k_l - 1, k_{l+1}, \dots, k_{r-1}, k_r - 1, \dots, k_{N-1}),$$

where the empty sum $\sum_{i=0}^{-1} k'_i$ should be understood as 0.

Note that, by definition, $\bar{G}(\zeta|\xi|x, y)$ reduces to the matrix element at $x = 0$:

$$\begin{aligned} \bar{G}(\zeta|\xi|0, y)_\mu^\epsilon &= \bar{F}(z|u|0) \\ &\times \sum_{i=0}^{N-1} \langle \Lambda_i | y^H \Phi_{\epsilon_1}(\zeta_1) \cdots \Phi_{\epsilon_m}(\zeta_m) \Psi_{\mu_n}^*(\xi_n) \cdots \Psi_{\mu_1}^*(\xi_1) | \Lambda_i \rangle. \end{aligned}$$

The matrix element can be calculated explicitly using the bosonization of the intertwining operators [10] on the Frenkel-Jing bosonization of $V(\Lambda_i)$ [4]. In fact the bosonization gives

the integral formula for the matrix elements. The integral of the extremal component can be calculated rather easily. For $N = 2$ such calculation is done in [8]. We give the integral formula in section 8 and the integrated formula with its derivation in section 9.

By specializing the formula of Proposition 3 in section 9 to $i = j$, $k_r = l_r$ ($0 \leq r \leq N - 1$), multiplying y_i and summing up in i from 0 to $N - 1$ we get

Proposition 2 *We have*

$$f(\zeta|\xi|0, y) = C \prod_{a=1}^n \left(\frac{\zeta_a}{\xi_a}\right)^{(N-1)(1-a)} \prod_{r=0}^{N-1} \prod_{a \in M_r^0} \left(\frac{\xi_a}{\zeta_a}\right)^r \\ \times \prod_{r < l} \prod_{a \in M_r^0, b \in M_l^0} \frac{(z_a - qu_b)(u_a - qz_b)}{(z_a - q^2z_b)(u_a - q^2u_b)} \sum_{i=0}^{N-1} y_i \prod_{a=1}^n \left(\frac{\zeta_a}{\xi_a}\right)^i \prod_{a=1}^{K_{i-1}} \left(\frac{u_a}{z_a}\right)$$

where $K_j = k_0 + \dots + k_j$, $K_{-1} = 0$, $y_0 = 1$,

$$C = (-1)^{\sum_{r=0}^{N-2} (r+1)k_r} q^{\frac{1}{2}K_{N-1}^2 - \frac{1}{2}\sum_{r=0}^{N-2} k_r^2 + K_{N-2}k_{N-1}}.$$

The empty product from 1 to 0 should be understood as one.

Proof of Theorem 1. It is sufficient to prove that $\det \left(f(\zeta_{M_0}, \dots, \zeta_{M_{N-1}} | \xi_{L_0}, \dots, \xi_{L_{N-1}} | 0, 1) \right)_{M,L}$ does not vanish identically, where 1 means $y_i = 1$ for any i . Let $P = \prod_{a=0}^{N-2} n - K_{a-1} C_{k_a}$ be the size of the determinant. We set

$$f_1(\zeta_1, \dots, \zeta_n) = \prod_{a=1}^n \zeta_a^{(N-1)(1-a)} \prod_{r=0}^{N-1} \prod_{a \in M_r^0} \zeta_a^{-r}, \quad f_2(\xi_1, \dots, \xi_n) = f_1(\xi_1, \dots, \xi_n)^{-1}.$$

Then by Proposition 2 we have

$$\det \left(f(\zeta_{M_0}, \dots, \zeta_{M_{N-1}} | \xi_{L_0}, \dots, \xi_{L_{N-1}} | 0, 1) \right)_{M,L} \\ = C^P \prod_M f_1(\zeta_{M_0}, \dots, \zeta_{M_{N-1}}) f_2(\xi_{M_0}, \dots, \xi_{M_{N-1}}) \\ \times \prod_M \prod_{r < l} \prod_{a \in M_r, b \in M_l} (z_a - q^2z_b)^{-1} (u_a - q^2u_b)^{-1} \mathcal{D}(\zeta|\xi|q),$$

where $\mathcal{D}(\zeta|\xi|q) = \det(\mathcal{D}(\zeta|\xi|q)_{M,L})_{M,L}$ and

$$\mathcal{D}(\zeta|\xi|q)_{M,L} \\ = \prod_{r < l} \left[\prod_{a \in M_r, b \in L_l} (z_a - qu_b) \prod_{a \in L_r, b \in M_l} (u_a - qz_b) \right] \sum_{j=0}^{N-1} \prod_{a=1}^n \left(\frac{\zeta_a}{\xi_a}\right)^j \prod_{r=0}^{j-1} \left(\prod_{a \in M_r} z_a^{-1} \prod_{a \in L_r} u_a \right).$$

We consider the case $q = 1$ and $\zeta_j = \xi_j$ for any j . It is easy to see that if M is different from L then $\mathcal{D}(\zeta|\xi|1)_{M,L} = 0$. Hence the matrix $(\mathcal{D}(\zeta|\xi|1)_{M,L})_{M,L}$ is a diagonal matrix and

$$\det(\mathcal{D}(\zeta|\xi|1)_{M,L})_{M,L} = \prod_M \mathcal{D}(\zeta|\xi|1)_{M,M} = N^P \prod_M \prod_{r < l} \prod_{a \in M_r, b \in M_l} (z_a - z_b)^2.$$

This completes the proof.

7 Determinant formula at $N = 2$ and $x = q^2$

Let us give here examples of the explicit formulae for the determinants of \bar{G} . To understand the general structure of the formula in the example below we first present the system of equations satisfied by the determinant and give one solution of it. By taking the determinant of the equation of \bar{G} at the weight space $(V^{\otimes n})_{k_0, k_1}$ we have

$$\begin{aligned} \left(\frac{\zeta_{i+1}}{\zeta_i}\right)^{nC_{k_0}} \left(\frac{z_i - q^2 z_{i+1}}{z_{i+1} - q^2 z_i}\right)^{n-2C_{k_0}-1} \det \bar{G}(\cdots \zeta_i \zeta_{i+1} \cdots | \xi | x, y) &= \det \bar{G}(\cdots \zeta_{i+1} \zeta_i \cdots | \xi | x, y), \\ \left(\frac{\xi_i}{\xi_{i+1}}\right)^{nC_{k_0}} \left(\frac{u_i - q^2 u_{i+1}}{u_{i+1} - q^2 u_i}\right)^{n-2C_{k_0}-1} \det \bar{G}(\zeta | \cdots \xi_i \xi_{i+1} \cdots | x, y) &= \det \bar{G}(\zeta | \cdots \xi_{i+1} \xi_i \cdots | x, y), \\ \det \bar{G}(x^{-1} \zeta_1, \zeta_2, \cdots, \zeta_n | \xi | x, y) &= \det \bar{G}(\zeta_2, \cdots, \zeta_n, \zeta_1 | \xi | x, y) (-1)^{(n-1)n-2C_{k_0}-1} \left(\prod_{j=1}^n \frac{\zeta_1}{\xi_j}\right)^{nC_{k_0}}, \\ \det \bar{G}(\zeta | x \xi_1, \xi_2, \cdots, \xi_n | x, y) &= \det \bar{G}(\zeta | \xi_2, \cdots, \xi_n, \xi_1 | x, y) (-1)^{(n-1)n-2C_{k_0}-1} \left(\prod_{j=1}^n \frac{\zeta_j}{\xi_1}\right)^{nC_{k_0}}, \end{aligned}$$

If $x = q^2$ and $y = 1$, one solution to these system of equations is given by

$$\begin{aligned} Q(\zeta | \xi) &= \\ \left(\prod_{j=1}^n \left(\frac{\xi_j}{\zeta_j}\right)^{n-1} \prod_{k < k'} \frac{z_{k'} - q^2 z_k}{u_k - q^2 u_{k'}}\right)^{n-2C_{k_0}-1} &\left(\prod_{j=1}^n \left(\frac{\xi_j}{\zeta_j}\right)^{j-1} \theta_{q^2}(-\prod_{j=1}^n \frac{\xi_j}{\zeta_j})\right)^{nC_{k_0}}. \end{aligned}$$

Any other meromorphic solution of the equation is given by multiplying a meromorphic function which is symmetric and q^2 periodic in ζ_i 's and ξ_i 's respectively to $Q(\zeta | \xi)$.

Example 1. We consider the case of $k_0 = 0$. The formula is from [8].

$$\bar{G}(\zeta_1, \cdots, \zeta_n | \xi_1, \cdots, \xi_n | x, y)_{- \cdots -} = (x^2)_\infty \prod_{j=1}^n \left(\frac{\xi_j}{\zeta_j}\right)^j \theta_x(-y \prod_{j=1}^n \frac{\zeta_j}{\xi_j}).$$

Example 2. Let us consider the case $x = q^2$ $y = 1$, $n = 2$, $k_0 = 1$. Then

$$\begin{aligned} \bar{G}(\zeta_1, \zeta_2 | \xi_1, \xi_2 | q^2, 1)_{+ -}^+ &= \\ q(q^4)_\infty \prod_{j=1}^2 \left(\frac{\xi_j}{\zeta_j}\right) \frac{\theta_{q^2}(-q \prod_{j=1}^2 \frac{\xi_j}{\zeta_j})}{u_1 - q^2 u_2} u_2 \left(1 - \frac{\zeta_1 \xi_1}{\zeta_2 \xi_2}\right), \end{aligned} \quad (24)$$

and

$$\begin{aligned} \det(\bar{G}(\zeta_1, \zeta_2 | \xi_1, \xi_2)_{\mu_1 \mu_2}^{\epsilon_1 \epsilon_2}) &= \\ -q^2 (q^4)_\infty^2 \prod_{j=1}^2 \left(\frac{\xi_j}{\zeta_j}\right)^{2j} \frac{z_2 - q^2 z_1}{u_1 - q^2 u_2} \theta_{q^2}(-q \prod_{j=1}^2 \frac{\xi_j}{\zeta_j})^2. \end{aligned} \quad (25)$$

Here $(z)_\infty = (z : q^4)_\infty$ and $z_i = \zeta_i^2$, $u_i = \xi_i^2$. This formula is calculated using the technique found in [12].

By using (22), (23) we can calculate other components of \bar{G} . By Dynkin diagram symmetry we know a priori that $\bar{G}(\zeta|\xi|q^2, 1)_{-+}^{++} = \bar{G}(\zeta|\xi|q^2, 1)_{+-}^{+-}$, $\bar{G}(\zeta|\xi|q^2, 1)_{-+}^{--} = \bar{G}(\zeta|\xi|q^2, 1)_{+-}^{--}$. Then the concrete expression for \bar{G} is given by

$$\begin{aligned} \bar{G}(\zeta|\xi|q^2, 1)(v_+ \otimes v_-) &= \\ (q^4)_\infty \prod_{j=1}^2 \left(\frac{\xi_j}{\zeta_j} \right) \frac{\theta_{q^2}(-q \prod_{j=1}^2 \frac{\xi_j}{\zeta_j})}{u_1 - q^2 u_2} u_2 &\left(q \left(1 - \frac{\zeta_1 \xi_1}{\zeta_2 \xi_2} \right) v_+ \otimes v_- - \frac{\xi_1}{\zeta_2} \left(1 - q^2 \frac{\zeta_1 \xi_2}{\zeta_2 \xi_1} \right) v_- \otimes v_+ \right), \\ \bar{G}(\zeta|\xi|q^2, 1)(v_- \otimes v_+) &= \\ (q^4)_\infty \prod_{j=1}^2 \left(\frac{\xi_j}{\zeta_j} \right) \frac{\theta_{q^2}(-q \prod_{j=1}^2 \frac{\xi_j}{\zeta_j})}{u_1 - q^2 u_2} u_2 &\left(- \frac{\xi_1}{\zeta_2} \left(1 - q^2 \frac{\zeta_1 \xi_2}{\zeta_2 \xi_1} \right) v_+ \otimes v_- + q \left(1 - \frac{\zeta_1 \xi_1}{\zeta_2 \xi_2} \right) v_- \otimes v_+ \right). \end{aligned}$$

8 Integral formula for matrix elements

We set

$$\bar{G}^{(ij)}(\zeta|\xi)_{\mu_1, \dots, \mu_n}^{\epsilon_1, \dots, \epsilon_m} = \frac{\langle \Lambda_i | \Phi_{\epsilon_1}(\zeta_1) \cdots \Phi_{\epsilon_m}(\zeta_m) \Psi_{\mu_n}^*(\xi_n) \cdots \Psi_{\mu_1}^*(\xi_1) | \Lambda_j \rangle}{\bar{F}(z|u|0)},$$

where $\bar{F}(z|u|0)$ is given by (14) and (15). We need to assume $j + n = i + m \bmod N$ for the matrix element to be well defined.

Let $k_r = \#\{j | \epsilon_j = r\}$, $l_r = \#\{j | \mu_j = r\}$ for $0 \leq j \leq N-1$. Then $m = \sum_{r=0}^{N-1} k_r$ and $n = \sum_{r=0}^{N-1} l_r$. The function $\bar{G}^{(ij)}(\zeta|\xi)_\mu^\epsilon$ is zero unless

$$\sum_{r=1}^n \text{wt} v_{\mu_r} + \Lambda_j = \sum_{r=1}^m \text{wt} v_{\epsilon_r} + \Lambda_i.$$

Since $\text{wt} v_r = \Lambda_{r+1} - \Lambda_r$ this condition is written as

$$k_r - l_r = k_{r-1} - l_{r-1} + \delta_{r,i} - \delta_{r,j} \quad 0 \leq r \leq N-1, \quad (26)$$

where we understand $k_{-1} = k_{N-1}$ and $l_{-1} = l_{N-1}$. In particular we have $m - n = j - i + Nr_0$, where $r_0 = k_{N-1} - l_{N-1}$. We assume the condition (26). We set $w_N^{(a)} = q^{N+1} z_a$ and $v_N^{(b)} = q^{N+1} u_b$ for the sake of convenience. Then

$$\begin{aligned} &\bar{G}^{(ij)}(\zeta|\xi)_{\mu_1, \dots, \mu_n}^{\epsilon_1, \dots, \epsilon_m} \\ &= \bar{C}^{(ij)}(\epsilon, \mu) \prod_{a=1}^m \zeta_a^{(N-1)(m-n+1-a)+j-\epsilon_a} \prod_{b=1}^n \xi_b^{(N-1)(b-1)-j+\mu_b} \\ &\times \int_{C_{\epsilon_1+1}^{(1)}} \frac{dw_{\epsilon_1+1}^{(1)}}{2\pi i} \cdots \int_{\bar{C}_{N-1}^{(n)}} \frac{dv_{N-1}^{(n)}}{2\pi i} \prod_{a; \epsilon_a \leq j-1} (w_j^{(a)})^{-1} \prod_{b; \mu_b \leq j-1} v_j^{(b)} \end{aligned}$$

$$\begin{aligned}
& \times \prod_{a=1}^m \prod_{k=\epsilon_a+1}^{N-1} \frac{(q^{-1}-q)w_k^{(a)}}{(w_k^{(a)}-q^{-1}w_{k+1}^{(a)})(w_k^{(a)}-qw_{k+1}^{(a)})} \prod_{b=1}^n \prod_{k=\mu_b+1}^{N-1} \frac{(q^{-1}-q)v_{k+1}^{(b)}}{(v_k^{(b)}-q^{-1}v_{k+1}^{(b)})(v_k^{(b)}-qv_{k+1}^{(b)})} \\
& \times \prod_{a<b} \prod_k \frac{-1}{w_k^{(a)}-qw_{k-1}^{(b)}} \prod_{a<b} \prod_k \frac{1}{w_k^{(a)}-qw_{k+1}^{(b)}} \prod_{a<b} \prod_{k \leq N-1} (w_k^{(a)}-q^2w_k^{(b)})(w_k^{(a)}-w_k^{(b)}) \\
& \times \prod_{a<b} \prod_k \frac{-1}{v_k^{(b)}-q^{-1}v_{k-1}^{(a)}} \prod_{a<b} \prod_k \frac{1}{v_k^{(b)}-q^{-1}v_{k+1}^{(a)}} \prod_{a<b} \prod_{k \leq N-1} (v_k^{(b)}-q^{-2}v_k^{(a)})(v_k^{(b)}-v_k^{(a)}) \\
& \times \prod_{a,b,k} (w_k^{(a)}-v_{k+1}^{(b)}) \prod_{a,b,k} (v_{k-1}^{(b)}-w_k^{(a)}) \prod_{a,b} \prod_{k \leq N-1} \frac{1}{(w_k^{(a)}-qv_k^{(b)})(w_k^{(a)}-q^{-1}v_k^{(b)})},
\end{aligned}$$

where

$$\begin{aligned}
\bar{C}^{(ij)}(\epsilon, \mu) &= (-1)^{\frac{1}{2} \sum_{a=1}^m (N-N\{\frac{i-1+a}{N}\})(N-1-N\{\frac{i-1+a}{N}\}) + \frac{1}{2} \sum_{b=1}^n (N-N\{\frac{j-1+b}{N}\})(N-1-N\{\frac{j-1+b}{N}\})} \\
& \times (-1)^{-ir_0(N-1) + \delta_{j0}(n+m)(N-1) + (\frac{1}{2}N(N+1)+1)(k_0+l_0)} \\
& \times (-1)^{\frac{1}{2}(j-i)(N-j)(N-j-1)\theta(1 \leq i < j) + \frac{1}{2}(i-j)(N-i)(N-1+i-2j)\theta(i > j \geq 1)} \\
& \times (-1)^{\frac{1}{2}(k_0+l_0) \sum_{r=1}^{N-1} (N-1-r)(N+2+r)(k_r+l_r)} \\
& \times q^{\frac{1}{2}(N+1)((i-j)(i-j-1) + r_0^2 N(N-1) + 2jr_0 N - 2ir_0(N-1))}.
\end{aligned}$$

Here, for a rational number r , we denote by $\{r\}$ the fractional part of r , that is, $\{r\} = r - [r]$, $[r]$ being the Gauss symbol. This notation appears only in the description of the sign and should not be confused with the double infinite product.

The integral variables are $w_k^{(a)}$ $a = 1, \dots, m$, $k = \epsilon_a + 1, \dots, N-1$ and $v_k^{(b)}$ $b = 1, \dots, n$, $k = \epsilon_b + 1, \dots, N-1$.

Each product in a, b, k which appears in the integrand is over all possible values satisfying the conditions written in the product symbol. We must be careful if $w_N^{(a)}$ or $v_N^{(b)}$ appears in the product. For example

$$\begin{aligned}
& \prod_{a<b} \prod_k \frac{1}{w_k^{(a)}-qw_{k+1}^{(b)}} \\
&= \prod_{1 \leq a < b \leq m-k_{N-1}} \prod_{k=\max(\epsilon_a+1, \epsilon_b)}^{N-2} \frac{1}{w_k^{(a)}-qw_{k+1}^{(b)}} \prod_{1 \leq a < b \leq m-k_{N-1}} \frac{1}{w_{N-1}^{(a)}-qw_N^{(b)}} \\
& \times \prod_{a=1}^{m-k_{N-1}} \prod_{b=m-k_{N-1}+1}^m \frac{1}{w_{N-1}^{(a)}-qw_N^{(b)}}.
\end{aligned}$$

This is because the index a of $w_k^{(a)}$ runs until $m-k_{N-1}$ if $k \leq N-1$, while a can run until m if $k = N$.

The integration contours $C_k^{(a)}$ of $w_k^{(a)}$ and $\tilde{C}_k^{(b)}$ of $v_k^{(b)}$ are as follows.

The contour $C_k^{(a)}$ is a simple closed curve going round the origin in the anticlockwise direction such that $qw_{k\pm 1}^{(a)}$, $q^{\pm 1}w_{k\pm 1}^{(b)}$ ($a < b$), $q^{\pm 1}v_k^{(b)}$ (any b) are inside, $q^{-1}w_{k\pm 1}^{(a)}$ and $q^{\pm 1}w_{k\pm 1}^{(b)}$ ($a > b$) are outside.

The contour $\tilde{C}_k^{(b)}$ is a simple closed curve going round the origin in the anticlockwise direction such that $q^{-1}v_{k\pm 1}^{(b)}$, $q^{\pm 1}v_{k\pm 1}^{(a)}$ ($a < b$) are inside, $qv_{k\pm 1}^{(b)}$, $q^{\pm 1}w_k^{(a)}$ (any a) and $q^{\pm 1}v_{k\pm 1}^{(a)}$ ($a > b$) are outside.

9 Integrated formula for the extremal component

In this section we give the formula without integration for the extremal component of the matrix element. For $0 \leq r \leq N-1$ we define K_r , L_r by

$$K_r = \sum_{r'=0}^r k_{r'}, \quad L_r = \sum_{r'=0}^r l_{r'}.$$

We set $K_r = L_r = 0$ for $r < 0$ or $r \geq N$. For a proposition P we define $\theta(P) = 1$ if P is true and $\theta(P) = 0$ otherwise. The variables are related by $z_r = \zeta_r^N$, $u_r = \xi_r^N$.

Proposition 3 *We have*

$$\begin{aligned} & \bar{G}^{(ij)}(\zeta|\xi)_{0^{k_0} \dots (N-1)^{k_{N-1}}}^{0^{l_0} \dots (N-1)^{l_{N-1}}} \\ &= C^{(ij)}(\mathbf{k}|\mathbf{l}) \prod_{a=1}^m \zeta_a^{(N-1)(m-n+1-a)-\epsilon_a+j} \prod_{b=1}^n \xi_b^{(N-1)(b-1)+\mu_b-j} \prod_{a=1}^{K_{j-1}} z_a^{-1} \prod_{b=1}^{L_{j-1}} u_b \\ & \quad \times \frac{\prod_{a=1}^m \prod_{b=1}^n (z_a - qu_b)^{\theta(\epsilon_a < \mu_b)} (u_b - qz_a)^{\theta(\epsilon_a > \mu_b)}}{\prod_{a,b=1}^m (z_a - q^2 z_b)^{\theta(\epsilon_a < \epsilon_b)} \prod_{a,b=1}^n (u_a - q^2 u_b)^{\theta(\mu_a < \mu_b)}}, \end{aligned}$$

where $C^{(ij)}(\mathbf{k}|\mathbf{l})$ is a constant given by

$$\begin{aligned} & C^{(ij)}(\mathbf{k}|\mathbf{l}) \\ &= (-1)^{\frac{1}{2} \sum_{a=1}^m (N-N\{\frac{i-1+a}{N}\})(N-1-N\{\frac{i-1+a}{N}\}) + \frac{1}{2} \sum_{b=1}^n (N-N\{\frac{j-1+b}{N}\})(N-1-N\{\frac{j-1+b}{N}\})} \\ & \quad \times (-1)^{-ir_0(N-1) + \sum_{r=0}^{N-2} (N-1-r)k_r + \delta_{j0}(n+m)(N-1) + (\frac{1}{2}N(N+1)+1)(k_0+l_0)} \\ & \quad \times (-1)^{\frac{1}{2}(j-i)(N-j)(N-j-1)\theta(1 \leq i < j) + \frac{1}{2}(i-j)(N-i)(N-1+i-2j)\theta(i > j \geq 1)} \\ & \quad \times (-1)^{\frac{1}{2}(k_0+l_0) \sum_{r=1}^{N-1} (N-1-r)(N+2+r)(k_r+l_r)} \\ & \quad \times (-1)^{\sum_{1 \leq a < b \leq K_{N-2}} (N-1-\epsilon_b) + \sum_{1 \leq a < b \leq L_{N-2}} (N-1-\mu_b) + \sum_{a=1}^{K_{N-2}} \sum_{b=1}^{L_{N-2}} (N-1-\max(\epsilon_a, \mu_b))} \\ & \quad \times q^{\frac{1}{2}(N+1)((i-j)(i-j-1)+r_0^2 N(N-1)+2jr_0 N-2ir_0(N-1)) + (j+1)(-K_{j-1}+L_{j-1})-N_{K_{N-2}} C_2 - (N-1)_{L_{N-2}} C_2} \\ & \quad \times q^{N(-K_{N-2}+L_{N-2})(k_{N-1}-l_{N-1}) + L_{N-2} l_{N-1} - \sum_{r=0}^{N-2} (N-1-r)k_r + \sum_{r=0}^{N-2} (r+1)_{k_r} C_2 + \sum_{r=0}^{N-2} r_{l_r} C_2} \\ & \quad \times q^{-\sum_{r=0}^{N-2} (r+1)k_r l_r + NK_{N-2} L_{N-2}}. \end{aligned} \tag{27}$$

and $\epsilon = (0^{k_0}, \dots, (N-1)^{k_{N-1}})$, $\mu = (0^{l_0}, \dots, (N-1)^{l_{N-1}})$.

Let us explain how to derive this formula. First we carry out the integration in the variable w by the order

$$w_1^{(1)} \rightarrow w_2^{(1)} \rightarrow \dots \rightarrow w_{N-1}^{(1)} \rightarrow w_1^{(2)} \rightarrow \dots \rightarrow w_{N-1}^{(K_{N-3}+1)} \rightarrow \dots \rightarrow w_{N-1}^{(K_{N-2})},$$

that is, first in $w_1^{(1)}$, next in $w_2^{(1)}$ etc. After the integration in w we integrate in the variable v by the order

$$v_1^{(1)} \rightarrow v_2^{(1)} \rightarrow \dots \rightarrow v_{N-1}^{(1)} \rightarrow v_1^{(2)} \rightarrow \dots \rightarrow v_{N-1}^{(K_{N-3}+1)} \rightarrow \dots \rightarrow v_{N-1}^{(K_{N-2})}.$$

In the variable $w_1^{(1)}$ the poles of the differential form in the integrand outside the contour $C_1^{(1)}$ is only at $w_1^{(1)} = q^{-1}w_2^{(1)}$. It means that there are no poles at infinity too. Thus we can calculate the integral in $w_1^{(1)}$ by taking the residue at $w_1^{(1)} = q^{-1}w_2^{(1)}$. After taking this residue the integrand have the same structure in the variable $w_2^{(1)}$ and so on. Therefore the integral in w 's is calculated by taking residues successively. After calculating the integral in the variables w the poles of the integrand in the variable $v_1^{(1)}$ inside the contour $\tilde{C}_1^{(1)}$ is only at $v_1^{(1)} = q^{-1}v_2^{(1)}$. Hence the integral is calculated by taking the residue at $v_1^{(1)} = q^{-1}v_2^{(1)}$. After taking the residue in $v_1^{(1)}$ the integrand has the same structure in the variable $v_2^{(1)}$ and so on.

Thus the integral is calculated by substituting

$$\begin{aligned} \frac{(q^{-1} - q)w_k^{(a)}}{(w_k^{(a)} - qw_{k+1}^{(a)})(w_k^{(a)} - q^{-1}w_{k+1}^{(a)})} &= q^{-1}, \quad w_k^{(a)} = q^{k+1}z_a, \\ \frac{(q^{-1} - q)v_{k+1}^{(b)}}{(v_k^{(b)} - qv_{k+1}^{(b)})(v_k^{(b)} - q^{-1}v_{k+1}^{(b)})} &= 1, \quad v_k^{(b)} = q^{k+1}u_b, \end{aligned}$$

into the integrand and multiplying it by

$$(-1)^{\sum_{r=0}^{N-1} (N-1-r)k_r}$$

which comes from taking the residue in w outside the contour.

Example. $m = n = 1$ case.

In this case $i = j$, $\epsilon_1 = \mu_1$, $k_r = l_r$ for any r and $r_0 = 0$. We consider the case $i = 0$. The integral formula is read as

$$\begin{aligned} \bar{G}^{(00)}(\zeta_1 | \xi_1)_\epsilon &= C^{(00)}(\epsilon | \epsilon) \zeta_1^{-\epsilon} \xi_1^\epsilon \times I, \\ I &= \int_{C_{\epsilon+1}^{(1)}} \frac{dw_{\epsilon+1}^{(1)}}{2\pi i} \dots \int_{C_{N-1}^{(1)}} \frac{dw_{N-1}^{(1)}}{2\pi i} \int_{\tilde{C}_{\epsilon+1}^{(1)}} \frac{dv_{\epsilon+1}^{(1)}}{2\pi i} \dots \int_{\tilde{C}_{N-1}^{(1)}} \frac{dv_{N-1}^{(1)}}{2\pi i} \\ &\quad \prod_{k=\epsilon+1}^{N-1} \frac{(q^{-1} - q)w_k^{(1)}}{(w_k^{(1)} - q^{-1}w_{k+1}^{(1)})(w_k^{(1)} - qw_{k+1}^{(1)})} \prod_{k=\epsilon+1}^{N-1} \frac{(q^{-1} - q)v_{k+1}^{(1)}}{(v_k^{(1)} - q^{-1}v_{k+1}^{(1)})(v_k^{(1)} - qv_{k+1}^{(1)})} \\ &\quad \times \prod_{k=\epsilon+1}^{N-1} (w_k^{(1)} - v_{k+1}^{(1)}) \prod_{k=\epsilon+2}^N (v_{k-1}^{(1)} - w_k^{(1)}) \prod_{k=\epsilon+1}^{N-1} \frac{1}{(w_k^{(1)} - qv_k^{(1)})(w_k^{(1)} - q^{-1}v_k^{(1)})}. \end{aligned}$$

Here, by calculation, $C^{(00)}(\epsilon|\epsilon) = 1$. Let us denote the integrand of I by J . Consider the integral in $w_{\epsilon+1}^{(1)}$. By definition of the contour $C_{\epsilon+1}^{(1)}$, $q^{-1}w_{\epsilon+2}^{(1)}$ are outside and all other poles on the complex plane are inside of $C_{\epsilon+1}^{(1)}$. The differential form $Jdw_{\epsilon+1}^{(1)}$ has no poles at ∞ . We have

$$\begin{aligned} \text{Res}_{w_{\epsilon+1}^{(1)}=q^{-1}w_{\epsilon+2}^{(1)}} Jdw_{\epsilon+1}^{(1)} = & \\ & - \prod_{k=\epsilon+2}^{N-1} \frac{(q^{-1}-q)w_k^{(1)}}{(w_k^{(1)}-q^{-1}w_{k+1}^{(1)})(w_k^{(1)}-qw_{k+1}^{(1)})} \prod_{k=\epsilon+1}^{N-1} \frac{(q^{-1}-q)v_{k+1}^{(1)}}{(v_k^{(1)}-q^{-1}v_{k+1}^{(1)})(v_k^{(1)}-qv_{k+1}^{(1)})} \\ & \times \prod_{k=\epsilon+2}^{N-1} (w_k^{(1)}-v_{k+1}^{(1)}) \prod_{k=\epsilon+3}^N (v_{k-1}^{(1)}-w_k^{(1)}) \prod_{k=\epsilon+3}^{N-1} \frac{1}{(w_k^{(1)}-qv_k^{(1)})(w_k^{(1)}-q^{-1}v_k^{(1)})} \\ & \frac{1}{(w_{\epsilon+2}^{(1)}-q^2v_{\epsilon+1}^{(1)})(w_{\epsilon+2}^{(1)}-q^{-1}v_{\epsilon+2}^{(1)})}. \end{aligned} \quad (28)$$

Consider this function (28) as a function of $w_{\epsilon+2}^{(1)}$. By the definition of the contour $C_{\epsilon+2}^{(1)}$, $q^{-1}w_{\epsilon+3}^{(1)}$ is outside of $C_{\epsilon+2}^{(1)}$ and all other poles on the complex plane are inside. The differential forms (28) $\times dw_{\epsilon+2}^{(1)}$ has no singularity at ∞ . Thus

$$\int_{C_{\epsilon+2}^{(1)}} (28)dw_{\epsilon+2}^{(1)} = -\text{Res}_{w_{\epsilon+2}^{(1)}=q^{-1}w_{\epsilon+3}^{(1)}} (28)dw_{\epsilon+2}^{(1)}$$

and so on. Consequently we have

$$\begin{aligned} & \int_{C_{\epsilon+1}^{(1)}} \frac{dw_{\epsilon+1}^{(1)}}{2\pi i} \cdots \int_{C_{N-1}^{(1)}} \frac{dw_{N-1}^{(1)}}{2\pi i} J \\ &= (-1)^{N-1-\epsilon} \text{Res}_{w_{N-1}^{(1)}=q^N z_1} \text{Res}_{w_{N-2}^{(1)}=q^{-1}w_{N-1}^{(1)}} \cdots \text{Res}_{w_{\epsilon+1}^{(1)}=q^{-1}w_{\epsilon+2}^{(1)}} Jdw_{\epsilon+1}^{(1)} \cdots dw_{N-1}^{(1)} \\ &= q^{\epsilon+1} \frac{z_1 - qu_i}{q^{\epsilon+1}z_1 - v_{\epsilon+1}^{(1)}} \prod_{k=\epsilon+1}^{N-1} \frac{(q^{-1}-q)v_{k+1}^{(1)}}{(v_k^{(1)}-q^{-1}v_{k+1}^{(1)})(v_k^{(1)}-qv_{k+1}^{(1)})}. \end{aligned} \quad (29)$$

A similar consideration is applicable to the function (29) in the variables v 's. Finally we have

$$\begin{aligned} I &= \text{Res}_{v_{N-1}^{(1)}=q^N u_1} \cdots \text{Res}_{v_{\epsilon+1}^{(1)}=q^{-1}v_{\epsilon+2}^{(1)}} (29)dv_{\epsilon+1}^{(1)} \cdots dv_{N-1}^{(1)} \\ &= 1. \end{aligned}$$

Consequently

$$\bar{G}^{(00)}(\zeta_1|\xi_1)_\epsilon = (\zeta_1^{-1}\xi_1)_\epsilon.$$

This reproduces the formula in [2].

10 Integral formula for the trace of intertwining operators

We recall the definition of $\bar{G}^{(i)}$:

$$\bar{G}^{(i)}(\zeta|\xi|x,y)_{\mu_1,\dots,\mu_n}^{\epsilon_1,\dots,\epsilon_m} = \frac{\text{tr}_{V(\Lambda_i)}(x^D y^H \Phi_{\epsilon_1}(\zeta_1) \cdots \Phi_{\epsilon_m}(\zeta_m) \Psi_{\mu_n}^*(\xi_n) \cdots \Psi_{\mu_1}^*(\xi_1))}{\bar{F}(z|u|x)},$$

where $\bar{F}(z|u|x)$ is given by (14) and (15).

We define

$$\bar{A}_r = \{j | \epsilon_j = r\}, \quad A_r = \bar{A}_0 \sqcup \cdots \sqcup \bar{A}_r, \quad \bar{B}_r = \{j | \mu_j = r\}, \quad B_r = \bar{B}_0 \sqcup \cdots \sqcup \bar{B}_r.$$

Then $\sharp \bar{A}_r = k_r$, $\sharp \bar{B}_r = l_r$ and $\sharp A_r = K_r$, $\sharp B_r = L_r$.

The condition that the weight, with respect to $U'_q(\widehat{\mathfrak{sl}}_N)$, of the composition of the intertwining operators are zero is

$$k_r - l_r = k_{N-1} - l_{N-1} =: r_0$$

for $0 \leq r \leq N-1$. We assume this condition. We set $(z)_\infty = (z; x^N)_\infty$, $z_a = \zeta_a^N$, $u_b = \xi_b^N$. Then for $0 \leq i \leq N-1$ we have

$$\begin{aligned} & \bar{G}^{(i)}(\zeta|\xi)_{\mu_1, \dots, \mu_n}^{\epsilon_1, \dots, \epsilon_m} \\ &= C^{tr(i)}(\epsilon|\mu) \prod_{a=1}^m \zeta_a^{(N-1)(m-n-a+1)-\epsilon_a+i} \prod_{b=1}^n \xi_b^{(N-1)(b-1)+\mu_b-i} \prod_{a \in A_{N-2}} z_a^{-1} \\ & \times \prod_{a < b, b \in A_{N-2}} z_a^{-1} \prod_{a < b, a \in B_{N-2}} u_b^{-1} \prod_{a \in A_{N-2}, k} \int_{C_k^{tr(a)}} \frac{dw_k^{(a)}}{2\pi i w_k^{(a)}} \prod_{b \in B_{N-2}, k} \int_{\bar{C}_k^{tr(b)}} \frac{dv_k^{(b)}}{2\pi i v_k^{(b)}} \\ & \times \prod_{a < b, a, b \in A_{N-2}} (w_{\epsilon_b+1}^{(a)})^{\theta(\epsilon_a \leq \epsilon_b)} (w_{\epsilon_b}^{(a)})^{-\theta(\epsilon_a < \epsilon_b)} \prod_{a < b, a, b \in B_{N-2}} (v_{\mu_a+1}^{(b)})^{\theta(\mu_a \geq \mu_b)} (v_{\mu_a}^{(b)})^{-\theta(\mu_a > \mu_b)} \\ & \times \prod_{a \in A_{N-2}, b \in B_{N-2}} (w_{\mu_b}^{(a)})^{\theta(\epsilon_a < \mu_b)} (v_{\epsilon_a}^{(b)})^{\theta(\epsilon_a > \mu_b)} \prod_{a \in A_{N-2}} (w_{N-1}^{(a)})^{l_{N-1}} \prod_{b \in B_{N-2}} (v_{N-1}^{(b)})^{k_{N-1}} \\ & \times \prod_{a < b, a \in A_{N-2}, b \in \bar{A}_{N-1}} (w_{N-1}^{(a)})^{-1} \prod_{a < b, b \in B_{N-2}, a \in \bar{B}_{N-1}} (v_{N-1}^{(b)})^{-1} \prod_{a \in A_{N-2}} w_{\epsilon_a+1}^{(a)} \\ & \times \prod_{a < b} \prod_k (1 - qw_k^{(a)}/w_{k+1}^{(b)}) \prod_{a > b} \prod_k (1 - qw_{k+1}^{(b)}/w_k^{(a)}) \prod_{a, b, k} \frac{1}{(qw_k^{(a)}/w_{k+1}^{(b)})_\infty (qw_{k+1}^{(b)}/w_k^{(a)})_\infty} \\ & \times \prod_{a < b} \prod_k (1 - q^{-1}v_{k+1}^{(b)}/v_k^{(a)}) \prod_{a > b} \prod_k (1 - q^{-1}v_k^{(a)}/v_{k+1}^{(b)}) \prod_{a, b, k} \frac{1}{(q^{-1}v_k^{(a)}/v_{k+1}^{(b)})_\infty (q^{-1}v_{k+1}^{(b)}/v_k^{(a)})_\infty} \\ & \times \prod_{a, b, k} \frac{\theta_{x^N}(v_{k+1}^{(b)}/w_k^{(a)})}{(x^N)_\infty} \prod_{a, b, k} \frac{\theta_{x^N}(w_{k+1}^{(a)}/v_k^{(b)})}{(x^N)_\infty} \prod_{a, b} \prod_{k \leq N-1} \frac{(x^N)_\infty^2}{\theta_{x^N}(qv_k^{(b)}/w_k^{(a)}) \theta_{x^N}(qw_k^{(a)}/v_k^{(b)})} \end{aligned}$$

$$\begin{aligned}
& \times \prod_{a < b} \prod_{k \leq N-1} \frac{\theta_{x^N}(w_k^{(b)}/w_k^{(a)})}{(x^N)_\infty} (q^2 x^N w_k^{(a)}/w_k^{(b)})_\infty (q^2 w_k^{(b)}/w_k^{(a)})_\infty \\
& \times \prod_{a < b} \prod_{k \leq N-1} \frac{\theta_{x^N}(v_k^{(a)}/v_k^{(b)})}{(x^N)_\infty} (q^{-2} v_k^{(a)}/v_k^{(b)})_\infty (q^{-2} x^N v_k^{(b)}/v_k^{(a)})_\infty \\
& \times (y_i \prod_{a: \epsilon_a + 1 \leq i} (w_i^{(a)})^{-1} \prod_{b: \mu_b + 1 \leq i} v_i^{(b)})^{1-\delta_{i0}} \theta_i(g_0^{-1} g_1^2 g_2^{-1}, \dots, g_{N-2}^{-1} g_{N-1}^2 g_N^{-1} | x^N),
\end{aligned}$$

where

$$\begin{aligned}
g_0^{-1} &= (-1)^{(m-n)(N-1)}, \quad g_N^{-1} = q^{(m-n)(N+1)} \prod_{a=1}^m z_a \prod_{b=1}^n u_b^{-1}, \\
g_j &= y_j \prod_{a: \epsilon_a + 1 \leq j} (w_j^{(a)})^{-1} \prod_{b: \mu_b + 1 \leq j} v_j^{(b)} \quad (1 \leq j \leq N-1), \\
\theta_i(z_1, \dots, z_{N-1} | p) &= \sum_{\alpha \in \bar{Q}} p^{\frac{1}{2}(\alpha|\alpha) + (\alpha|\Lambda_i)} \prod_{j=1}^{N-1} z_j^{(\alpha|\Lambda_j)}.
\end{aligned}$$

Here $\bar{Q} = \mathbf{Z}\alpha_1 \oplus \dots \oplus \mathbf{Z}\alpha_{N-1}$ is the root lattice of sl_N . The constant $C^{tr(i)}(\epsilon|\mu)$ is given by

$$\begin{aligned}
& C^{tr(i)}(\epsilon|\mu) \\
&= (-1)^{ir_0(N-1) + \text{sgn}_N(i) + \frac{1}{3}r_0(N-1)(N-2)(N-3) + \frac{1}{2}r_0(r_0+1)(N-1) + K_{N-2} + nL_{N-2}} \\
& \times (-1)^{\sum_{a=1}^{N-2} ak_a + \sum_{a \in A_{N-2}} a + \sum_{b \in B_{N-2}} b + \sum_{a \in A_{N-2}, b \in B_{N-2}} c_{ab}} \\
& \times (-1)^{\sum_{a < b, a, b \in A_{N-2}} (\epsilon_{ab} + \theta(\epsilon_a \leq \epsilon_b)) + \sum_{a < b, a, b \in B_{N-2}} (\mu_{ab} + \theta(\mu_a \geq \mu_b))} \\
& \times q^{ir_0(N+1) + \frac{1}{2}r_0^2 N(N-1)(N+1) - n(N+1)L_{N-2} + (N-1)K_{N-2}L_{N-2} + \sum_{b=1}^{N-2} bl_b} \\
& \times q^{(N+1)(-\sum_{a \in A_{N-2}} a + \sum_{b \in B_{N-2}} b) - \sum_{a \in A_{N-2}, b \in B_{N-2}} c_{ab}} \\
& \times \left(\frac{\{q^2 x^N\}}{\{q^{2N} x^N\}} \right)^m \left(\frac{\{x^N\}}{\{q^{2N-2} x^N\}} \right)^n (x^N)_\infty^{\sum_{a=0}^{N-2} (N-1-a)(k_a + l_a) - 1} \\
& \times (q^2)_\infty^{\sum_{a=0}^{N-2} (N-1-a)k_a} (q^{-2})_\infty^{\sum_{b=0}^{N-2} (N-1-b)l_b},
\end{aligned}$$

where we set

$$\begin{aligned}
\epsilon_{ab} &= \max(\epsilon_a, \epsilon_b), \quad \mu_{ab} = \max(\mu_a, \mu_b), \quad c_{ab} = \max(\epsilon_a, \mu_b), \\
\text{sgn}_N(i) &= \\
& \frac{1}{2} \sum_{a=1}^m (N - N\{\frac{i+a-1}{N}\})(N-1 - N\{\frac{i+a-1}{N}\}) \\
& + \frac{1}{2} \sum_{b=1}^n (N - N\{\frac{i+b-1}{N}\})(N-1 - N\{\frac{i+b-1}{N}\}).
\end{aligned}$$

The integration contour $C_k^{tr(a)}$ for $w_k^{(a)}$ and $\tilde{C}_k^{tr(b)}$ for $w_k^{(b)}$ are specified in the following manner.

The contour $C_k^{tr(a)}$ is a simple closed curve going round the origin in the anticlockwise direction such that

$qx^{Nm}w_{k\pm 1}^{(b)}$ ($m \geq 0$, any b), $x^{Nm}w_k^{(b)}$ ($m \geq 1$, $b \neq a$), $x^{Nm}v_{k\pm 1}^{(b)}$ ($m \geq 1$, any b), $q^{-1}x^{Nm}v_k^{(b)}$ ($m \geq 0$, any b), $q^{N+1}x^{Nm}u_b$ ($m \geq 1$, any b), $q^{N+2}x^{Nm}z_b$ ($m \geq 0$, any b) are inside and $q^{-1}x^{-Nm}w_{k\pm 1}^{(b)}$ ($m \geq 0$, any b), $x^{-Nm}w_k^{(b)}$ ($m \geq 1$, $b \neq a$), $x^{-Nm}v_{k\pm 1}^{(b)}$ ($m \geq 1$, any b), $qx^{-Nm}v_k^{(b)}$ ($m \geq 1$, any b), $q^{N+1}x^{-Nm}u_b$ ($m \geq 1$, any b), $q^Nx^{-Nm}z_b$ ($m \geq 0$, any b) are outside.

The contour $\tilde{C}_k^{tr(b)}$ is a simple closed curve going round the origin in the anticlockwise direction such that

$q^{-1}x^{Nm}v_{k\pm 1}^{(a)}$ ($m \geq 0$, any a), $q^{-2}x^{Nm}v_k^{(b)}$ ($m \geq 1$, $b \neq a$), $q^{N+1}x^{Nm}z_a$ ($m \geq 1$, any a), $q^Nx^{Nm}u_a$ ($m \geq 0$, any a), $x^{Nm}w_{k\pm 1}^{(a)}$ ($m \geq 1$, any a), $q^{-1}x^{Nm}w_k^{(a)}$ ($m \geq 1$, any a) are inside and $qx^{-Nm}v_{k\pm 1}^{(a)}$ ($m \geq 0$, any a), $q^2x^{-Nm}v_k^{(b)}$ ($m \geq 1$, $b \neq a$), $q^{N+1}x^{-Nm}z_a$ ($m \geq 1$, any a), $q^{N+2}x^{-Nm}u_a$ ($m \geq 1$, any a), $x^{-Nm}w_{k\pm 1}^{(a)}$ ($m \geq 1$, any a), $qx^{-Nm}w_k^{(a)}$ ($m \geq 1$, any a) are outside.

It can be checked that those contours are well defined for $|x| < |q|^{2/N} < 1$.

If we set $x = 0$ in the formula above, we obtain the integral formula for the matrix element with $i = j$ in section 8.

We have verified that this formula coincides with the trace formula in [8] for $N = 2$.

The derivation of the integral formula of the trace is totally similar to the sl_2 case [9][8] and it is briefly explained in the appendix.

As a corollary of the integral formula for the trace we have

Corollary 2 *The functions $G(\zeta|\xi|x, \mathbf{y})$ and $\bar{G}(\zeta|\xi|x, \mathbf{y})$ are meromorphic functions on $(\mathbf{C}^*)^{n+m}$, where $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$ is the algebraic torus.*

Proof. The singularity of the integral appears only when the pinch of the integration contour occurs. By the definition of the contour the pinch happens at $X = q^ax^bY$ for some integers a, b , where $X, Y \in \{\zeta_1, \dots, \zeta_m, \xi_1, \dots, \xi_n\}$. Suppose that pinch occurs at $X = q^ax^bY$. We decompose the integral into the sum of residues and the integral with the integration contour for which the pinch does not occur at $X = q^ax^bY$. Since the integrand of the trace formula is a meromorphic function on $(\mathbf{C}^*)^{n+m}$ its residue is also a meromorphic function on $(\mathbf{C}^*)^{n+m}$. In the decomposition the singularity at $X = q^ax^bY$ appears only from the residue part. Thus the singularity of the trace function at $X = q^ax^bY$ is a pole.

11 Discussion

In this paper we have proved that the trace of the composition of the intertwining operators of type I and type II gives a basis of the solution space of the qKZ equation at generic values of parameters. The qKZ equation considered in this paper takes the value in the tensor product of the finite dimensional irreducible $U_q(\mathfrak{sl}_N)$ module with the highest weight Λ_1 .

There is a problem whether it is possible to construct solutions of the qKZ equation taking values in the tensor product of the arbitrary finite dimensional irreducible $U_q(\mathfrak{sl}_N)$ modules as a

trace of intertwining operators. For $N = 2$ it will be possible to construct solutions of the qKZ equation taking values in the arbitrary irreducible $U_q(sl_2)$ modules by taking the trace of the intertwining operators introduced in [11] over the tensor product of integrable highest weight $U_q(\widehat{sl}_2)$ modules of level one [7]. It is natural to expect that the trace functions thus constructed give a basis of the solution space. For $N \geq 3$ a similar construction will be possible. For the moment what kind of modules we can treat is not very clear.

Let us consider the qKZ equation (1) of $N = 2$ on the weight subspace of $V_1 \otimes \cdots \otimes V_n$ with a weight, say λ . At some special values of κ , which depend on p, q and λ , the hypergeometric solution of Tarasov-Varchenko [15] takes the value in the space of singular vectors with respect to certain action of $U_q(sl_2)$. From the experience of rational and level zero case [12], it is probable that the trace function still gives a basis of the full space of the tensor product at those special values of κ . This means that the hypergeometric solution and the trace solution have very different structures. It is an interesting and important problem to relate these two basis. A partial result in this direction is given in [12].

One of the important properties of the trace construction of the solution is that it gives a map from $V^{\otimes n}$ with fixed n to the space of solutions of the qKZ equation taking the value in $V^{\otimes m}$ for any m . This will be a key structure to relate finite and infinite dimensional modules. Note that it is nothing but the typical structure of the form factors in integrable quantum field theories [13][12]. The above mentioned problem connecting two types of solutions is also important to understand the completeness problem of local fields constructed by Smirnov [13][1].

The value $x = q^2$ is of particular interest, since the correlation functions and the form factors of the solvable lattice model are given by some special case of the trace function at this value of x . We conjecture that $\det \bar{G}$ does not vanish identically at $x = q^2$.

The generalization of the results in this paper to other types of quantum affine algebra is also interesting.

The trace of intertwining operators are also studied in [3]. Here we simply comments the following things. In [3] the trace is twisted by the Dynkin diagram automorphism and thus it is different from the trace considered in this paper. The difference equations satisfied by the trace in [3] and in this paper are also different.

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A Integral formula for trace $-U_q(\widehat{sl}_2)$ case-

In the case of $U_q(\widehat{sl}_2)$ the integral formula for the trace is given in [8]. Our formula at $N = 2$ in section 10 recovers it. In this case the sum of the trace over $V(\Lambda_0)$ and $V(\Lambda_1)$ simplifies a bit. It is used in the calculation of the example in section 7. Thus we shall present this simplified formula. It also serves as a simplest example of the trace formula.

$$\bar{G}(\zeta|\xi|x, y)_{\mu_1, \dots, \mu_n}^{\epsilon_1, \dots, \epsilon_m} =$$

$$C_{st}^{mn} \prod_{j=1}^m \zeta_j^{-j+(1+\epsilon_j)/2} \prod_{k=1}^n \xi_k^{k-(1+\mu_k)/2} \prod_{r=1}^s \int_{\mathcal{C}} \frac{dw_r}{2\pi i w_r} \prod_{r=1}^t \int_{\tilde{\mathcal{C}}} \frac{dv_r}{2\pi i v_r} F^{AB}(\zeta, \xi, w, v|x, y),$$

where

$$\begin{aligned} F^{AB}(\zeta, \xi, w, v|x, y) = & \prod_{r=1}^s \left(\prod_{j < a_r} (qz_j - q^{-1}w_r) \prod_{j > a_r} (z_j - w_r) \right) \prod_{j,r} \frac{1}{(w_r/z_j)_{\infty} (q^2 z_j/w_r)_{\infty}} \\ & \times \prod_{r=1}^t \left(\prod_{k < b_r} (qv_r^{-1} - q^{-1}u_k^{-1}) \prod_{k > b_r} (v_r^{-1} - u_k^{-1}) \right) \prod_{k,r} \frac{1}{(q^{-2}v_r/u_k)_{\infty} (u_k/v_r)_{\infty}} \\ & \times \prod_{r,j} \frac{\theta_{x^2}(-q^{-1}v_r/z_j)}{(x^2)_{\infty}} \prod_{r,k} \frac{\theta_{x^2}(-qu_k/w_r)}{(x^2)_{\infty}} \\ & \times \prod_{r,r'} \frac{(x^2)_{\infty}^2}{\theta_{x^2}(-qv_r/w_{r'}) \theta_{x^2}(-q^{-1}v_r/w_{r'})} \\ & \times \prod_{r < r'} \frac{(q^2 w_r/w_{r'})_{\infty} (q^2 w_{r'}/w_r)_{\infty} w_{r'}^{-1} \theta_{x^2}(w_{r'}/w_r)}{w_{r'} - q^2 w_r (x^2)_{\infty}} \\ & \prod_{r < r'} (v_{r'} - q^{-2}v_r) (x^2 q^{-2} v_r/v_{r'})_{\infty} (x^2 q^{-2} v_{r'}/v_r)_{\infty} \frac{v_{r'} \theta_{x^2}(v_r/v_{r'})}{(x^2)_{\infty}} \\ & \times \theta_x((-1)^{t+1} (-q)^{\frac{m-n}{2}} y \frac{\prod \zeta_j \prod v_r}{\prod \xi_k \prod w_r}). \end{aligned}$$

Here $A = \{a_1 < \dots < a_s\} = \{j | \epsilon_j = +\}$ and $B = \{b_1 < \dots < b_t\} = \{j | \mu_j = +\}$, $z_j = \zeta_j^2$, $u_k = -\xi_k^2$. The integral contour \mathcal{C} and $\tilde{\mathcal{C}}$ go round the origin such that for \mathcal{C} : $q^2 x^{2l} z_j$ ($l \geq 0$) are inside and $x^{-2l} z_j$ ($l \geq 0$) are outside, for $\tilde{\mathcal{C}}$: $x^{2l} u_k$ ($l \geq 0$) are inside and $q^2 x^{-2l} u_k$ ($l \geq 0$) are outside, $-q^{\pm 1} x^{2l} w_r$ ($l \geq 1$) are inside and $-q^{\pm 1} x^{-2l} w_r$ ($l \geq 0$) are outside.

We have rewritten the formula in [8] using the following formula:

$$\theta_{x^4}(-xX^2) + (-1)^t X \theta_{x^4}(-x^3 X^2) = \theta_x((-1)^{t+1} X).$$

B Boson expression of intertwining operators

Here we recall the bosonic expression of intertwining operators for $U_q(\widehat{sl_N})$ [10].

Let us consider the Heisenberg algebra generated by $\{a_i(k) | 1 \leq i \leq N-1, k \in \mathbf{Z} \setminus \{0\}\}$ with the commutation relation

$$[a_i(k), a_j(l)] = \delta_{k+l,0} \frac{[(\alpha_i | \alpha_j) k][k]}{k}.$$

Let \mathcal{H} be the Fock space of this algebra, $\mathcal{H} = \mathbf{C}[a_i(-k) | 1 \leq i \leq N-1, k \in \mathbf{Z} \setminus \{0\}]$. Let $\bar{Q} = \bigoplus_{j=1}^{N-1} \mathbf{Z} \alpha_j$ be the root lattice of sl_N . Then the twisted group algebra $\mathbf{C}[\bar{Q}]$ is the algebra generated by $e^{\alpha_1}, \dots, e^{\alpha_{N-1}}$ with the defining relation

$$e^{\alpha_i} e^{\alpha_j} = (-1)^{(\alpha_i | \alpha_j)} e^{\alpha_j} e^{\alpha_i}.$$

Then

Theorem 2 [4] *There is an isomorphism*

$$V(\Lambda_i) \simeq \mathcal{H} \otimes \mathbf{C}[\bar{Q}]e^{\Lambda_i}, \quad (30)$$

where $\mathcal{H} \otimes \mathbf{C}[\bar{Q}]e^{\Lambda_i}$ is the vector space consisting of the symbols Xe^{Λ_i} , $X \in \mathbf{C}[\bar{Q}]$.

For the action of the generators of $U_q(\widehat{\mathfrak{sl}_N})$ on the right hand side of (30) see [4][10].

To describe the intertwining operators we introduce the algebra containing $\mathbf{C}[\bar{Q}]e^{\Lambda_i}$. Let $\bar{P} = \oplus_{j=1}^{N-1} \mathbf{Z}\Lambda_j = \oplus_{j=2}^{N-1} \mathbf{Z}\alpha_j \oplus \mathbf{Z}\Lambda_{N-1}$ be the weight lattice of sl_N . Then the extended group algebra $\mathbf{C}[\bar{P}]$ is the algebra generated by $e^{\alpha_1}, \dots, e^{\alpha_{N-1}}, e^{\Lambda_{N-1}}$ with the defining relation [10]

$$e^\alpha e^\beta = (-1)^{(\alpha|\beta)} e^\beta e^\alpha, \quad \alpha, \beta \in \{\alpha_2, \dots, \alpha_{N-1}, \Lambda_{N-1}\}.$$

As a convention, for $\alpha = \sum_{j=2}^{N-1} m_j \alpha_j + m_N \Lambda_{N-1}$, we set

$$e^\alpha = e^{m_2 \alpha_2} \dots e^{m_{N-1} \alpha_{N-1}} e^{m_N \Lambda_{N-1}}.$$

Note that

$$\alpha_1 = - \sum_{r=2}^{N-1} r \alpha_r + N \Lambda_{N-1}, \quad \Lambda_i = - \sum_{r=i+1}^{N-1} (r-i) \alpha_r + (N-i) \Lambda_{N-1}.$$

The algebra $\mathbf{C}[\bar{Q}]$ becomes a subalgebra of $\mathbf{C}[\bar{P}]$. We consider $\mathbf{C}[\bar{Q}]e^{\Lambda_i}$ as a subspace of $\mathbf{C}[\bar{P}]$.

We define the action of the symbols ∂_α , e^α ($\alpha \in \bar{Q}$), and d on the space $\mathcal{H} \otimes \mathbf{C}[\bar{Q}]e^{\Lambda_i}$. Let $X = a_{j_1}(-n_1) \dots a_{j_k}(-n_k) \in \mathcal{H}$, $e^\beta \in \mathcal{H} \otimes \mathbf{C}[\bar{Q}]e^{\Lambda_i}$ and $Y = X \otimes e^\beta$. Then

$$\begin{aligned} \partial_\alpha Y &= (\alpha|\beta)Y, \quad e^\alpha Y = X \otimes e^\alpha e^\beta, \\ dY &= \left(-\sum_{r=1}^k n_k - \frac{(\beta|\beta)}{2} + \frac{(\Lambda_i|\Lambda_i)}{2}\right)Y. \end{aligned}$$

Then the principal grading operator $D^{(i)}$ on $V(\Lambda_i)$ is given by

$$D^{(i)} = -\rho + \frac{i(N-i)}{2}, \quad \rho = Nd + \frac{1}{2} \sum_{r=1}^{N-1} r(N-r) \partial_{\alpha_r}.$$

We set

$$\begin{aligned} X_j^\pm(w) &= \exp\left(\pm \sum_{k=1}^{\infty} \frac{a_j(-k)}{[k]} q^{\mp \frac{k}{2}} w^k\right) \exp\left(\mp \sum_{k=1}^{\infty} \frac{a_j(k)}{[k]} q^{\mp \frac{k}{2}} w^{-k}\right) e^{\pm \alpha_j} w^{\pm \partial_{\alpha_j}}, \\ &= \sum_{n \in \mathbf{Z}} x_{j,n}^\pm w^{-n-1}, \\ x_j^\pm &= x_{j,0}^\pm. \end{aligned}$$

Then

Theorem 3 [10]

$$\begin{aligned}
\tilde{\Phi}_{N-1}^{h(i)}(z) &= \exp\left(\sum_{k=1}^{\infty} a_{N-1}^*(-k) q^{(N+\frac{3}{2})k} z^k\right) \exp\left(\sum_{k=1}^{\infty} a_{N-1}^*(k) q^{-(N+\frac{1}{2})k} z^{-k}\right) \\
&\times e^{\Lambda_{N-1}} (q^{N+1} z)^{\partial_{\Lambda_{N-1}} + \frac{N-1-i}{N}} (-1)^{(N-1)(\partial_{\Lambda_1} - \frac{N-1-i}{N})} (-1)^{\frac{1}{2}(N-i)(N-1-i)}, \\
\tilde{\Phi}_j^{h(i)}(z) &= [\tilde{\Phi}_{j+1}^{h(i)}(z), x_j^-]_q, \quad 0 \leq j \leq N-2, \\
\tilde{\Psi}_{N-1}^{*h(i)}(u) &= \exp\left(-\sum_{k=1}^{\infty} a_{N-1}^*(-k) q^{(N+\frac{1}{2})k} u^k\right) \exp\left(-\sum_{k=1}^{\infty} a_{N-1}^*(k) q^{-(N+\frac{3}{2})k} u^{-k}\right) \\
&\times e^{-\Lambda_{N-1}} (q^{N+1} u)^{-\partial_{\Lambda_{N-1}} + \frac{i}{N}} (-1)^{(N-1)(-\partial_{\Lambda_1} + \frac{N-i}{N})} (-1)^{\frac{1}{2}(N-i)(N-1-i)}, \\
\tilde{\Psi}_{N-1}^{*h(i)}(u) &= [x_j^+, \tilde{\Psi}_{j+1}^{*h(i)}(u)]_{q^{-1}}, \quad 0 \leq j \leq N-2,
\end{aligned}$$

where $[X, Y]_q = XY - qYX$ and

$$a_{N-1}^*(k) = \frac{-1}{[k][Nk]} \sum_{r=1}^{N-1} [rk] a_r(k).$$

The elements $a_{N-1}^*(k)$ satisfy the relations

$$[a_j(k), a_{N-1}^*(-l)] = \delta_{k,l} \delta_{j,N-1} \frac{[k]}{k}, \quad [a_{N-1}^*(k), a_{N-1}^*(-l)] = -\delta_{k,l} \frac{q^k(1 - q^{(2N-2)k})}{k(1 - q^{2Nk})}.$$

The inner product is given explicitly by

$$(\Lambda_i | \Lambda_j) = \frac{i(N-j)}{N} \quad (i \leq j), \quad (\alpha_i | \Lambda_j) = \delta_{i,j}.$$

C List of normal ordering rules

We define the normal ordered operator as an operator of the form

$$\begin{aligned}
&\exp\left(\sum_{j=1}^{N-1} \sum_{n=1}^{\infty} A_n^{(j)} a_J(-n)\right) \exp\left(\sum_{j=1}^{N-1} \sum_{n=1}^{\infty} B_n^{(j)} a_J(n)\right) \\
&\times \exp\left(\sum_{j=1}^{N-1} c_j \alpha_j\right) \exp\left(\sum_{j=1}^{N-1} c'_j \partial_{\alpha_j}\right)
\end{aligned}$$

Thus we define the normal order of the product of operators as

$$\begin{aligned}
:a_i(k) a_j(l): &= a_i(k) a_j(l) \quad \text{if } k \leq l \\
&= a_j(l) a_i(k) \quad \text{if } k > l, \\
:\partial_{\alpha} a_i(k): &= :a_i(k) \partial_{\alpha} := a_i(k) \partial_{\alpha}, \\
:e^{\alpha} a_i(k): &= :a_i(k) e^{\alpha} := a_i(k) e^{\alpha}, \\
:\partial_{\alpha} e^{\beta}: &= :e^{\beta} \partial_{\alpha} := e^{\beta} \partial_{\alpha}.
\end{aligned}$$

We shall give a list of expressions of operators in terms of their normal ordered operators.

$$X_{j_1}^-(w_1)X_{j_2}^-(w_2) =: X_{j_1}^-(w_1)X_{j_2}^-(w_2) : \quad |j_1 - j_2| > 1, j_1, j_2 \neq 0,$$

$$X_1^-(w_1)X_j^-(w_2) = (-1)^{j+1} : X_1^-(w_1)X_j^-(w_2) : \quad j \geq 3,$$

$$X_j^-(w_1)X_1^-(w_2) = (-1)^{j-1} : X_j^-(w_1)X_1^-(w_2) : \quad j \geq 3,$$

$$X_j^-(w_1)X_{j+1}^-(w_2) = \frac{(-1)^{\delta_{j1}}}{w_1 - qw_2} : X_j^-(w_1)X_{j+1}^-(w_2) :,$$

$$X_{j+1}^-(w_1)X_j^-(w_2) = \frac{(-1)^{1-\delta_{j1}}}{w_1 - qw_2} : X_{j+1}^-(w_1)X_j^-(w_2) :,$$

$$X_j^-(w_1)X_j^-(w_2) = (w_1 - q^2w_2)(w_1 - w_2) : X_j^-(w_1)X_j^-(w_2) :,$$

$$X_{j_1}^-(w)X_{j_2}^+(v) =: X_{j_1}^-(w)X_{j_2}^+(v) : \quad |j_1 - j_2| > 1, j_1, j_2 \neq 1,$$

$$X_1^-(w_1)X_j^+(w_2) = (-1)^{j+1} : X_1^-(w_1)X_j^+(w_2) : \quad j \geq 3,$$

$$X_j^-(w_1)X_1^+(w_2) = (-1)^{j-1} : X_j^-(w_1)X_1^+(w_2) : \quad j \geq 3,$$

$$X_j^-(w)X_{j+1}^+(v) = (-1)^{\delta_{j1}}(w - v) : X_j^-(w)X_{j+1}^+(v) :,$$

$$X_{j+1}^-(w)X_j^+(v) = (-1)^{1-\delta_{j1}}(w - v) : X_{j+1}^-(w)X_j^+(v) :,$$

$$X_j^-(w)X_j^+(v) = \frac{1}{(w - qv)(w - q^{-1}v)} : X_j^-(w)X_j^+(v) :,$$

$$X_{j_1}^+(v)X_{j_2}^-(w) =: X_{j_1}^+(v)X_{j_2}^-(w) : \quad |j_1 - j_2| > 1, j_1, j_2 \neq 1,$$

$$X_1^+(w_1)X_j^-(w_2) = (-1)^{j+1} : X_1^+(w_1)X_j^-(w_2) : \quad j \geq 3,$$

$$X_j^+(w_1)X_1^-(w_2) = (-1)^{j-1} : X_j^+(w_1)X_1^-(w_2) : \quad j \geq 3,$$

$$X_j^+(v)X_{j+1}^-(w) = (-1)^{\delta_{j1}}(v - w) : X_j^+(v)X_{j+1}^-(w) :,$$

$$X_{j+1}^+(v)X_j^-(w) = (-1)^{1-\delta_{j1}}(v - w) : X_{j+1}^+(v)X_j^-(w) :,$$

$$X_j^+(v)X_j^-(w) = \frac{1}{(v - qw)(v - q^{-1}w)} : X_j^+(v)X_j^-(w) :,$$

$$X_{j_1}^+(v_1)X_{j_2}^+(v_2) =: X_{j_1}^+(v_1)X_{j_2}^+(v_2) : \quad |j_1 - j_2| > 1, j_1, j_2 \neq 1,$$

$$X_1^+(w_1)X_j^+(w_2) = (-1)^{j+1} : X_1^+(w_1)X_j^+(w_2) : \quad j \geq 3,$$

$$X_j^+(w_1)X_1^+(w_2) = (-1)^{j-1} : X_j^+(w_1)X_1^+(w_2) : \quad j \geq 3,$$

$$X_j^+(v_1)X_{j+1}^+(v_2) = \frac{(-1)^{\delta_{j1}}}{v_1 - q^{-1}v_2} : X_j^+(v_1)X_{j+1}^+(v_2) :,$$

$$X_{j+1}^+(v_1)X_j^+(v_2) = \frac{(-1)^{1-\delta_{j1}}}{v_1 - q^{-1}v_2} : X_{j+1}^+(v_1)X_j^+(v_2) :,$$

$$X_j^+(v_1)X_j^+(v_2) = (v_1 - q^{-2}v_2)(v_1 - v_2) : X_j^+(v_1)X_j^+(v_2) :,$$

$$\tilde{\Phi}_{N-1}^{h(i)}(z)X_j^-(w) =: \tilde{\Phi}_{N-1}^{h(i)}(z)X_j^-(w) : \quad j \neq N-1,$$

$$\tilde{\Phi}_{N-1}^{h(i)}(z)X_{N-1}^-(w) = \frac{q^{-1}}{w - q^N z} : \tilde{\Phi}_{N-1}^{h(i)}(z)X_{N-1}^-(w) :,$$

$$\tilde{\Phi}_{N-1}^{h(i)}(z)X_j^+(v) =: \tilde{\Phi}_{N-1}^{h(i)}(z)X_j^+(v) : \quad j \neq N-1,$$

$$\tilde{\Phi}_{N-1}^{h(i)}(z)X_{N-1}^+(v) = (v - q^{N+1}z) : \tilde{\Phi}_{N-1}^{h(i)}(z)X_{N-1}^+(v) :,$$

$$\tilde{\Psi}_{N-1}^{*h(i)}(u)X_j^+(v) =: \tilde{\Psi}_{N-1}^{*h(i)}(u)X_j^+(v) : \quad j \neq N-1,$$

$$\tilde{\Psi}_{N-1}^{*h(i)}(u)X_{N-1}^+(v) = \frac{q}{v - q^{N+2}u} : \tilde{\Psi}_{N-1}^{*h(i)}(u)X_{N-1}^+(v) :,$$

$$X_j^-(w)\tilde{\Phi}_{N-1}^{h(i)}(z) =: X_j^-(w)\tilde{\Phi}_{N-1}^{h(i)}(z) :, \quad j \neq N-1,$$

$$X_{N-1}^-(w)\tilde{\Phi}_{N-1}^{h(i)}(z) = \frac{1}{w - q^{N+2}z} : X_{N-1}^-(w)\tilde{\Phi}_{N-1}^{h(i)}(z) :,$$

$$X_j^+(v)\tilde{\Phi}_{N-1}^{h(i)}(z) =: X_j^+(v)\tilde{\Phi}_{N-1}^{h(i)}(z) :, \quad j \neq N-1,$$

$$X_{N-1}^+(v)\tilde{\Phi}_{N-1}^{h(i)}(z) = (v - q^{N+1}z) : X_{N-1}^+(v)\tilde{\Phi}_{N-1}^{h(i)}(z) :,$$

$$X_j^+(v)\tilde{\Psi}_{N-1}^{*h(i)}(u) =: X_j^+(v)\tilde{\Psi}_{N-1}^{*h(i)}(u) :, \quad j \neq N-1,$$

$$X_{N-1}^+(v)\tilde{\Psi}_{N-1}^{*h(i)}(u) = \frac{1}{v - q^N u} : X_{N-1}^+(v)\tilde{\Psi}_{N-1}^{*h(i)}(u) :,$$

$$X_j^-(w)\tilde{\Psi}_{N-1}^{*h(i)}(u) = : X_j^-(w)\tilde{\Psi}_{N-1}^{*h(i)}(u) :, \quad j \neq N-1,$$

$$X_{N-1}^-(w)\tilde{\Psi}_{N-1}^{*h(i)}(u) = (w - q^{N+1}u) : X_{N-1}^-(w)\tilde{\Psi}_{N-1}^{*h(i)}(u) :,$$

$$\tilde{\Phi}_{N-1}^{h(i_1)}(z_1)\tilde{\Phi}_{N-1}^{h(i_2)}(z_2) = (-q^{N+1}z_1)^{\frac{N-1}{N}} \frac{(q^2 \frac{z_2}{z_1})_\infty}{(q^{2N} \frac{z_2}{z_1})_\infty} : \tilde{\Phi}_{N-1}^{h(i_1)}(z_1)\tilde{\Phi}_{N-1}^{h(i_2)}(z_2) :,$$

$$\tilde{\Phi}_{N-1}^{h(i_1)}(z)\tilde{\Psi}_{N-1}^{*h(i_2)}(u) = (-q^{N+1}z)^{-\frac{N-1}{N}} \frac{(q^{2N-1} \frac{u}{z})_\infty}{(q \frac{u}{z})_\infty} : \tilde{\Phi}_{N-1}^{h(i_1)}(z)\tilde{\Psi}_{N-1}^{*h(i_2)}(u) :,$$

$$\tilde{\Psi}_{N-1}^{*h(i_1)}(u)\tilde{\Phi}_{N-1}^{h(i_2)}(z) = (-q^{N+1}u)^{-\frac{N-1}{N}} \frac{(q^{2N-1} \frac{z}{u})_\infty}{(q \frac{z}{u})_\infty} : \tilde{\Psi}_{N-1}^{*h(i_1)}(u)\tilde{\Phi}_{N-1}^{h(i_2)}(z) :,$$

$$\tilde{\Psi}_{N-1}^{*h(i_2)}(u_2)\tilde{\Psi}_{N-1}^{*h(i_1)}(u_1) = (-q^{N+1}u_2)^{\frac{N-1}{N}} \frac{(\frac{u_1}{u_2})_\infty}{(q^{2N-2} \frac{u_1}{u_2})_\infty} : \tilde{\Psi}_{N-1}^{*h(i_2)}(u_2)\tilde{\Psi}_{N-1}^{*h(i_1)}(u_1) :.$$

Let us set

$$\begin{aligned} \tilde{\Phi}_j^{h(i)}(z|w_{N-1} \cdots w_{j+1}) &= [\tilde{\Phi}_{j+1}^{h(i)}(z|w_{N-1} \cdots w_{j+2}), X_{j+1}^-(w_{j+1})]_q, \\ \tilde{\Psi}_j^{*h(i)}(u|v_{N-1} \cdots v_{j+1}) &= [X_{j+1}^+(v_{j+1}), \tilde{\Psi}_{j+1}^{*h(i)}(u|v_{N-1} \cdots v_{j+2})]_{q^{-1}}. \end{aligned}$$

Then

$$\begin{aligned} &\tilde{\Phi}_j^{h(i)}(z|w_{N-1} \cdots w_{j+1}) \\ &= (-1)^{\frac{1}{2}N(N+1)\delta_{j0} + \delta_{j0}} \prod_{k=j+1}^{N-1} \frac{(q^{-1} - q)w_k}{(w_k - q^{-1}w_{k+1})(w_k - qw_{k+1})} \\ &\times : \tilde{\Phi}_{N-1}^{h(i)}(z)X_{N-1}^-(w_{N-1}) \cdots X_{j+1}^-(w_{j+1}) :, \end{aligned}$$

$$\begin{aligned} &\tilde{\Psi}_j^{*h(i)}(u|v_{N-1} \cdots v_{j+1}) \\ &= (-1)^{\frac{1}{2}N(N+1)\delta_{j0} + \delta_{j0}} \prod_{k=j+1}^{N-1} \frac{(q^{-1} - q)v_{k+1}}{(v_k - q^{-1}v_{k+1})(v_k - qv_{k+1})} \\ &\times : \tilde{\Psi}_{N-1}^{*h(i)}(u)X_{N-1}^+(v_{N-1}) \cdots X_{j+1}^+(v_{j+1}) :, \end{aligned}$$

where we set $w_N = q^{N+1}z$ and $v_N = q^{N+1}u$.

We have

$$: \tilde{\Phi}_{N-1}^{h(i_1)}(z_1)X_{N-1}^-(w_{N-1}^{(1)}) \cdots X_{j_1+1}^-(w_{j_1+1}^{(1)}) : : \tilde{\Phi}_{N-1}^{h(i_2)}(z_2)X_{N-1}^-(w_{N-1}^{(2)}) \cdots X_{j_2+1}^-(w_{j_2+1}^{(2)}) :$$

$$\begin{aligned}
&= (-1)^{\frac{1}{2}(N-1-j_2)(N+2+j_2)\delta_{j_1 0} + \frac{1}{2}(N-1-j_1)(N+2+j_1)\delta_{j_2 0}} \\
&\quad \times h^{(+)}\left(\frac{z_2}{z_1}\right) q^{-1} (-q^{N+1} z_1)^{\frac{N-1}{N}} \frac{\prod_k (w_k^{(1)} - q^2 w_k^{(2)}) (w_k^{(1)} - w_k^{(2)})}{(w_{N-1}^{(1)} - q^{N+2} z_2) (w_{N-1}^{(2)} - q^N z_1)} \\
&\quad \times \prod_k \frac{-1}{w_k^{(1)} - q w_{k-1}^{(2)}} \prod_k \frac{1}{w_k^{(1)} - q w_{k+1}^{(2)}} : \tilde{\Phi}_{N-1}^{h(i_1)}(z_1) \cdots X_{j_2+1}^-(w_{j_2+1}^{(2)}) :, \\
&: \tilde{\Phi}_{N-1}^{h(i_1)}(z) X_{N-1}^-(w_{N-1}) \cdots X_{j_1+1}^-(w_{j_1+1}) :: \tilde{\Psi}_{N-1}^{*h(i_2)}(u) X_{N-1}^+(v_{N-1}) \cdots X_{j_2+1}^+(v_{j_2+1}) : \\
&= (-1)^{\frac{1}{2}(N-1-j_2)(N+2+j_2)\delta_{j_1 0} + \frac{1}{2}(N-1-j_1)(N+2+j_1)\delta_{j_2 0}} \\
&\quad \times h^{(0)}\left(\frac{u}{z}\right)^{-1} (-q^{N+1} z)^{-\frac{N-1}{N}} \frac{(w_{N-1} - q^{N+1} u) (v_{N-1} - q^{N+1} z)}{\prod_k (w_k - q v_k) (w_k - q^{-1} v_k)} \\
&\quad \times \prod_k (w_k - v_{k+1}) \prod_k (-1) (w_k - v_{k-1}) : \tilde{\Phi}_{N-1}^{h(i_1)}(z) \cdots X_{j_2+1}^+(v_{j_2+1}) :, \\
&: \tilde{\Psi}_{N-1}^{*h(i_2)}(u_2) X_{N-1}^+(v_{N-1}^{(2)}) \cdots X_{j_2+1}^+(v_{j_2+1}^{(2)}) :: \tilde{\Psi}_{N-1}^{*h(i_1)}(u_1) X_{N-1}^+(v_{N-1}^{(1)}) \cdots X_{j_1+1}^+(v_{j_1+1}^{(1)}) : \\
&= (-1)^{\frac{1}{2}(N-1-j_2)(N+2+j_2)\delta_{j_1 0} + \frac{1}{2}(N-1-j_1)(N-2+j_1)\delta_{j_2 0}} \\
&\quad \times h^{(-)}\left(\frac{u_1}{u_2}\right) q (-q^{N+1} u_2)^{\frac{N-1}{N}} \frac{\prod_k (v_k^{(2)} - q^{-2} v_k^{(1)}) (v_k^{(2)} - v_k^{(1)})}{(v_{N-1}^{(1)} - q^{N+2} u_2) (v_{N-1}^{(2)} - q^N u_1)} \\
&\quad \times \prod_k \frac{1}{v_k^{(2)} - q^{-1} v_{k+1}^{(1)}} \prod_k \frac{-1}{v_k^{(2)} - q^{-1} v_{k-1}^{(1)}} : \tilde{\Psi}_{N-1}^{*h(i_2)}(u_2) \cdots X_{j_1+1}^+(v_{j_1+1}^{(1)}) : .
\end{aligned}$$

Here, denoting $(z)_\infty = (z; x^N)_\infty$, we set

$$h^{(+)}(z) = \frac{(q^2 z)_\infty}{(q^{2N} z)_\infty}, \quad h^{(0)}(z)^{-1} = \frac{(q^{2N-1} z)_\infty}{(q z)_\infty}, \quad h^{(-)}(z) = \frac{(z)_\infty}{(q^{2N-2} z)_\infty}.$$

D Derivation of integral formula for trace

The calculation of the trace using the bosonic expression of the intertwining operators are exactly similar to the case of sl_2 [9][8]. Thus we present here only the necessary information for the calculation of the trace.

We use the following formula

$$\begin{aligned}
& \text{tr}_{V(\Lambda_i)} \left(x^D \exp \left(\sum_{j=1}^{N-1} \sum_{n=1}^{\infty} A_n^{(j)} a_J(-n) \right) \exp \left(\sum_{j=1}^{N-1} \sum_{n=1}^{\infty} B_n^{(j)} a_J(n) \right) \right. \\
& \times \exp \left(\sum_{j=1}^{N-1} c_j \alpha_j \right) \prod_{j=1}^{N-1} g_j^{\partial_{\alpha_j}} g_0^{-\partial_{\Lambda_1}} g_N^{-\partial_{\Lambda_{N-1}}} \Big) \\
& = (x^N)_{\infty}^{-1} g_0^{-(\Lambda_1|\Lambda_i)} g_N^{-(\Lambda_{N-1}|\Lambda_i)} g_i^{1-\delta_{i0}} \theta_i(g_0^{-1} g_1^2 g_2^{-1}, \dots, g_{N-2}^{-1} g_{N-1}^2 g_N^{-1} | x^N) \\
& \times \prod_{j=1}^{N-1} \exp \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n} x^{Nmn} A_n^{(j)} (-[n]^2 B_n^{(j-1)} + [n][2n] B_n^{(j)} - [n]^2 B_n^{(j+1)}) \right), \quad (31)
\end{aligned}$$

where we set $B_n^{(0)} = B_n^{(N)} = 0$. The derivation of this formula is similar to the sl_2 case. We refer to [8] for details.

In the previous section we have given the expression of the operators in terms of their normally ordered operators. Therefore in this section we shall give a list of contributions to the trace from the normally ordered operators. Then using the formula (31) we can calculate the trace and the result is presented in section 10.

For an operator \mathcal{O} such that

$$\begin{aligned}
\mathcal{O} &= \exp \left(\sum_{j=1}^{N-1} \sum_{n=1}^{\infty} A_n^{(j)} a_J(-n) \right) \exp \left(\sum_{j=1}^{N-1} \sum_{n=1}^{\infty} B_n^{(j)} a_J(n) \right) \\
&\times \exp \left(\sum_{j=1}^{N-1} c_j \alpha_j \right) \prod_{j=1}^{N-1} g_j^{\partial_{\alpha_j}} g_0^{-\partial_{\Lambda_1}} g_N^{-\partial_{\Lambda_{N-1}}},
\end{aligned}$$

if we write

$$\mathcal{O} \approx J$$

then it means that

$$J = \prod_{j=1}^{N-1} \exp \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n} x^{Nmn} A_n^{(j)} (-[n]^2 B_n^{(j-1)} + [n][2n] B_n^{(j)} - [n]^2 B_n^{(j+1)}) \right).$$

The following is the list which is necessary for the calculation of the trace.

$$\tilde{\Phi}_{N-1}^{h(i)}(z) \approx \frac{\{q^2 x^N\}}{\{q^{2N} x^N\}}, \quad \tilde{\Psi}_{N-1}^{*h(i)}(u) \approx \frac{\{x^N\}}{\{q^{2N-2} x^N\}},$$

$$X_j^-(w) \approx (q^2 x^N)_{\infty} (x^N)_{\infty}, \quad X_j^+(w) \approx (q^{-2} x^N)_{\infty} (x^N)_{\infty}$$

$$: X_{j_1}^{\sigma_1}(w_1) X_{j_2}^{\sigma_2}(w_2) : \approx 1 \quad |j_1 - j_2| > 1, \sigma_1, \sigma_2 \in \{\pm\},$$

$$: X_j^-(w_1) X_{j+1}^-(w_2) : \approx \frac{1}{(q x^N w_1 / w_2)_{\infty} (q x^N w_2 / w_1)_{\infty}},$$

$$: X_j^-(w_1)X_j^-(w_2) : \approx (q^2 x^N w_1/w_2)_\infty (q^2 x^N w_2/w_1)_\infty (x^N w_1/w_2)_\infty (x^N w_2/w_1)_\infty,$$

$$: X_j^-(w)X_{j+1}^+(v) : \approx (x^N w/v)_\infty (x^N v/w)_\infty,$$

$$: X_{j+1}^-(w)X_j^+(v) : \approx (x^N w/v)_\infty (x^N v/w)_\infty,$$

$$: X_j^-(w)X_j^+(v) : \approx \frac{1}{(qx^N w/v)_\infty (qx^N v/w)_\infty (q^{-1}x^N w/v)_\infty (q^{-1}x^N v/w)_\infty},$$

$$: X_j^+(v_1)X_{j+1}^+(v_2) : \approx \frac{1}{(q^{-1}x^N v_1/v_2)_\infty (q^{-1}x^N v_2/v_1)_\infty},$$

$$: X_j^+(v_1)X_j^+(v_2) : \approx (x^N v_1/v_2)_\infty (x^N v_2/v_1)_\infty (q^{-2}x^N v_1/v_2)_\infty (q^{-2}x^N v_2/v_1)_\infty,$$

$$: \tilde{\Phi}_{N-1}^{h(i)}(z)X_j^\pm(w) : \approx 1, \quad j \neq N-1,$$

$$: \tilde{\Psi}_{N-1}^{*h(i)}(u)X_j^\pm(w) : \approx 1, \quad j \neq N-1,$$

$$: \tilde{\Phi}_{N-1}^{h(i)}(z)X_{N-1}^-(w) : \approx \frac{1}{(q^{N+2}x^N z/w)_\infty (q^{-N}x^N w/z)_\infty},$$

$$: \tilde{\Phi}_{N-1}^{h(i)}(z)X_{N-1}^+(v) : \approx (q^{N+1}x^N z/v)_\infty (q^{-N-1}x^N v/z)_\infty,$$

$$: \tilde{\Psi}_{N-1}^{*h(i)}(u)X_{N-1}^-(w) : \approx (q^{N+1}x^N u/w)_\infty (q^{-N-1}x^N w/u)_\infty,$$

$$: \tilde{\Psi}_{N-1}^{*h(i)}(u)X_{N-1}^+(v) : \approx \frac{1}{(q^N x^N u/v)_\infty (q^{-N-2}x^N v/u)_\infty},$$

$$: \tilde{\Phi}_{N-1}^{h(i_1)}(z_1)\tilde{\Phi}_{N-1}^{h(i_2)}(z_2) : \approx \frac{\{q^2 x^N z_1/z_2\}\{q^2 x^N z_2/z_1\}}{\{q^{2N} x^N z_1/z_2\}\{q^{2N} x^N z_2/z_1\}},$$

$$: \tilde{\Phi}_{N-1}^{h(i_1)}(z)\tilde{\Psi}_{N-1}^{*h(i_2)}(u) : \approx \frac{\{q^{2N-1} x^N z/u\}\{q^{2N-1} x^N u/z\}}{\{qx^N z/u\}\{qx^N u/z\}},$$

$$: \tilde{\Psi}_{N-1}^{*h(i_1)}(u_1)\tilde{\Psi}_{N-1}^{*h(i_2)}(u_2) : \approx \frac{\{x^N u_1/u_2\}\{x^N u_2/u_1\}}{\{q^{2N-2} x^N u_1/u_2\}\{q^{2N-2} x^N u_2/u_1\}}.$$

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